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ESTRATTO

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Expected loss rate

EDIZIONI DJSGE

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ABSTRACT

In letteratura sono ben noti i risultati relativi sulla condizione di incollaggio regolare alla base dell'esercizio anticipato ottimale delle opzioni. È facile dimostrare che un incollaggio regolare implica l'uguaglianza del tasso interno di rendimento tra l'opzione e la posizione di indebitamento. Il nostro scopo è studiare il tasso di perdita atteso nel caso di un esercizio non ottimale il costo per un comportamento non ottimale.

In litterature the relative results about the smooth pasting condition behind optimal early exercise of options are well known. It is easy to show that smooth pasting implies rate of return equalization between the option and the levered position that results from exercise. Our aim is to study the expected loss rate for non optimal exercise (the cost for non optimal behaviour).

PAROLE CHIAVE

Condizioni di contatto – Tassi di rendimento – Opzione elasticità.

Smooth pasting – Rates of return – Option elasticity

SOMMARIO: 1. Introduction. – 2. Rates of return. – 3. Expected loss rate. – 4. Conclusion.

1. The smooth pasting condition associated with option has generated considerable interest because of the optimality of early exercise. It is well known that smooth pasting is a first-order condition for optimum. it was proposed by Samuelson[1], Merton[2] and several others. Brekke and Øksendal[3] also show that the condition is sufficient under weak constraints. Nonetheless, smooth pasting remains somewhat mysterious to both economists and practitioners and it is apparently not very useful for many users up to theorists. Dixit et al.[4] found a link between theory and practice using an analogy

* Saggio sottoposto a referaggio secondo il sistema per *peer review*.
[1] P.A. Samuelson, *Rational theory of warrant pricing,* in *Management Review* 6(2), 1965, pp. 13-31.
[2] R.C. Merton, *Theory of rational option pricing*, in *The Bell Journal of Economics and Management Science* 4, 1973, pp. 141–183.
[3] K.A. Brekke, B. Øksendal, *The high contact principle as a sufficiency condition for optimal stopping* D. Lund, B. Øksendal (Eds.), *Stochastic Models and Option Values*, North-Holland, 1991, pp. 187-208.
[4] A. Dixit, R.S. Pindyck, S. Sødal, *A markup interpretation of optimal investment rules*, in *Economic Journal* 109 (455), 1999, pp. 179-189.

between optimal exercise of investment options of the McDonald and Siegel[5] model and application of standard market power models. Optimal investment can be characterized by an elasticity-based premium.
Shackleton and Sødal[6] provide another natural, explanation of the phenomenon; that of *rate of return equalization* between the option and its levered payoff. This idea allows to know and implement smooth pasting techniques in a wider variety of situations. We also relate results to the elasticity-based rules introduced by Dixit et al.[7] and Sødal[8] . The results are illustrated here using geometric Brownian motion but are also valid for other stochastic process. In this paper we want to study the expected loss rate for non optimal exercise ie the cost for non optimal behavior.
This paper is organized as follow. After the introduction, we model the rates of return in the next section, then we explain the goal of the work and obtain the main equation of the dynamics of power prices.

2. In this section we recall results known in literature[9] but useful for understanding the objective of the work.
It is well known that the Geometric Brownian diffusions can be written in the Risk Neutral Q or Real World P; with drift $r - q$ or $\mu - q$, where (r is the the rate of return risk free, q *is* the dividend, μ is the project of return and σ the volatility rates).
We consider the classic Black and Scholes[10] model, and we remember that the single risky asset is called the stock, whose price-per-share X_t follows the geometric Brownian motion:

$$dX_t = (\mu - q)X_t dt + \sigma X_t dW_t^P \qquad X_0 = x > 0 \tag{1}$$

$$dX_t = (r - q)X_t dt + \sigma X_t dW_t^Q \qquad X_0 = x > 0 \tag{2}$$

where the volatility $\sigma > 0$ and the expected rate of return $r \in R$ are constants. The process $W = \{W_t\, ;\, 0 \le t < +\infty\}$ is a standard Brownian motion on a probability space (Ω, F, P); let us denote $v(X_t)$ put price and by applying Itô's lemma to the function v we get the changes $dv(X_t)$ in a perpetual American put option as follow:

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5 R. McDonald, D. Siegel, *The value of waiting to invest,* in *Quarterly Journal of Economics*, 101, 1986, pp. 707-728.
6 M.B. Shackleton, S. Sødal, *Smooth pasting as rate of return equalization*, in *Economics Letters*, 89, 2005, pp. 200-206.
7 A. Dixit, R.S. Pindyck, S. Sødal, *A markup interpretation of optimal investment rules*, in *Economic Journal*, 109 (455), 1999, pp. 179-189.
8 S. Sødal, *A simplified exposition of smooth pasting* in *Economics Letters* 58, 1998, pp. 217-233.
9 M.B. Shackleton, S. Sødal, *Smooth pasting as rate of return equalization* in *Economics Letters*, 89, 2005, pp. 200-206.
10 F. Black, M. Scholes, *The Pricing of options and corporate liabilities*, in *Journal Political Economy*, 81, 1973, pp. 637–659.

$$dv(X_t) = \left(\tfrac{1}{2}\sigma^2 X_t^2 v''(X_t) + \mu X_t v'(X_t)\right)dt + \sigma X_t v'(X_t) dW_t. \tag{3}$$

No arbitrage condition requires that risk neutral expectation $\mathbb{E}^Q[dv(X_t)]$ of these changes must be risk free (or the hedged position yields the risk free rate):

$$\mathbb{E}^Q[dv(X_t)] = \left(\tfrac{1}{2}\sigma^2 X_t^2 v''(X_t) + (r-q)X_t v'(X_t)\right)dt \tag{4}$$

the real world expectation operator $\mathbb{E}^P[dv(X_t)]$ is:

$$\mathbb{E}^P[dv(X_t)] = \left(\tfrac{1}{2}\sigma^2 X_t^2 v''(X_t) + (\mu-q)X_t v'(X_t)\right)dt \tag{5}$$

and real word returns depend on the premium $\mu - r$ through the expectation operator $\mathbb{E}^P[dv(X_t)]$.
At this end we add and subtract $rX_t v'(X_t)$ in (5) so that:

$$\mathbb{E}^P[dv(X_t)] = (\mu - r)X_t v'(X_t)dt + rv(X_t)dt + \mathcal{L}v(X_t)\, dt \tag{6}$$

where:

$$\mathcal{L}v(X_t) := \frac{1}{2}\sigma^2 X_t^2 v''(X_t) + (r-q)X_t v'(X_t) - rv(X_t) \tag{7}$$

is the Black & Scholes operator.
We consider the local expected return rate of the put option:

$$r_v(X_t) = \frac{1}{dt}\frac{\mathbb{E}^P[dv(X_t)]}{v(X_t)} \tag{8}$$

By substituting the (6) and (7) in (8) we have:

$$r_v(X_t) = r + \mathcal{E}_v(X_t)(\mu - r) + \frac{\mathcal{L}v(X_t)}{v(X_t)} \tag{9}$$

where $\mathcal{E}_v = \frac{X_t v'(X_t)}{v(X_t)}$ and the elasticity $\mathcal{E}_v$ is interpreted as the relative beta[11].

[11] R.C. Merton, *Theory of rational option pricing* in *The Bell Journal of Economics and Management Science*, 4, 1973, pp. 141-183.

In the continuation region we have $\mathcal{L}v(X_t) = 0$ so that the expected return rate of the put option is given by:

$$r_v(X_t) = r + \mathcal{E}_v(X_t)(\mu - r) \tag{10}$$

In this case we know from Karatzas and Shreve[12], Karatzas[13] and Karatzas and Wang[14] that the optimal hedging portfolio weight, that is the proportion of the wealth that is invested in stock, can be expressed in terms of the elasticity of the value function v as $X_t v'(X_t)/v(X_t)$. Since this proportion is negative, it can be interpreted as the leverage ratio. Then $r_v(X_t)$ is also the local expected rate of return of the optimal hedging portfolio.

Let us consider a portfolio θ_t (the levered position that results from exercise at *t*) that invests $X_t g'(X_t)$ in stock and $g(X_t) - X_t g'(X_t)$ in bond. At the exercise time *t*, the value of this portfolio is $\theta_t = g(X_t)$ and its return rate is:

$$r_\theta(X_t) = \frac{rg(X_t) + X_t g'(X_t)(\mu - r)}{g(X_t)} \tag{11}$$

Therefore, the return rate of the levered portfolio θ_t (as a fraction of the payoff value) can be expressed as

$$r_\theta(X_t) = r + \mathcal{E}_g(X_t)(\mu - r) \tag{12}$$

where $\mathcal{E}_g(X_t)$ is the elasticity of *g* defined as:

$$\mathcal{E}_g(x) := \frac{x\, g'(x)}{g(x)} \qquad x \in]0, K[.$$

3. In this section we present the expected loss rate. We deal with perpetual American put options which have payoff at time t given by
$\Psi_t = g(X_t)$,

where the function $g: [0 + \infty[\longrightarrow \mathbb{R}$ is the payoff function and it is a decreasing continuous function with $g(0) > 0$ and $g(+\infty) = 0$.

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[12] I. Karatzas, S. Shreve, *Methods of mathematical finance*, SpringerVerlag, New York 1998.
[13] I. Karatzas, *Lectures on the mathematics finance. CRM Monograph Series 8*, American Mathematical Society, 1996.
[14] I. Karatzas, H. Wang, *A barrier option of American type*, in *Applied Mathematics and Optimization*, 42(3), 2000, pp. 259-279.

Let $K := sup\{x \geq 0;\ g(x) > 0\}$ and let us suppose that g is differentiable on the interval $[0; K[$. We can prove that if $K < +\infty$ then it is the contracted strike price. Shackeleton and Sødal[15] show that smooth pasting condition implies (at the point of optimal put exercise) rate of return equalization between the option and the levered position that results from exercise. The expected rate of return of the option is equal to the (expected) return rate of the levered payoff.
Then, at smooth pasting, the expected loss rate (defined as the difference between the expected return rate of the option and the (expected) return rate of the levered payoff) is equal to zero when the option is optimally exercised.
Our aim is to study the expected loss rate for non-optimal exercise (the cost for non-optimal behavior).
(This is a measure of returns, but measured per unit of x instead of time). It makes sense to consider the loss rate only when the option is in the money ie when $0 < x < K$.
The corresponding expected loss rate is given by:

$$l(X_t) = r_\theta(X_t) - r_v(X_t) \tag{13}$$

From (9) and (11) we have:

$$l(X_t) = (\mathcal{E}_g(X_t) - \mathcal{E}_v(X_t))\,(\mu - r) - \frac{\mathcal{L}v(X_t)}{v(X_t)} \tag{14}$$

We call b the unique solution of the equation

$$\frac{x\,g'(x)}{g(x)} = -\frac{2r}{\sigma^2}$$

In the exercise region $(X_t < b)$ we have $v = g$, $v \in C^2$, and then

$$l(X_t) = \frac{\mathcal{L}g(X_t)}{g(X_t)}$$

In the continuation region $(X_t > b)$ we have $\mathcal{L}v(X_t) = 0$ and so formula (14) become:

$$l(X_t) = (\mathcal{E}_g(X_t) - \mathcal{E}_v(X_t))\,(\mu - r).$$

[15] M.B. Shackleton, S. Sødal, *Smooth pasting as rate of return equalization*, in *Economics Letters*, 89, 2005, pp. 200-206.

At optimal exercise ie $X_t = b$ otherwise we have:

$$l(b^-) = -\frac{\mathcal{L}g(b)}{g(b)}$$

And the smooth pasting condition implies:

$l(b^+) = 0.$

We observe that the expected loss rate verifies these conditions:

1. $l(x) < 0$ when $b < x < K$ (since the elasticity of *g* is strictly decreasing), and,
2. $l(x) > 0$ when $0 < x < b$ (since the elasticity of *g* is strictly decreasing that implies $\mathcal{L}g(x)$<0).

Moreover, we have:

$$l(x) = r_\theta\,(x) - r_v(x) = R_\theta\,(x) - R_v(x)$$

the expected loss rate is the difference between the excess expected return rate of the perpetual American put option $R_v(x)$ and $R_\theta\,(x)$ that is the excess expected return rate of the levered portfolio θ .
where:

$$R_v(x) = r_v(x) - r =$$

$$= \begin{cases} \mathcal{E}_v(x)(\mu - r) = -\frac{2r}{\sigma^2}(\mu - r), & b \le x < K \\ \left(\mathcal{E}_g(x) - \frac{c_t(x)}{g(x)(\mu - r)}\right)(\mu - r), & 0 < x < b \end{cases}$$

is the excess expected return (rate) of the perpetual American put option[16],
And $R_\theta\,(x)$ is the excess expected return (rate) of the levered portfolio θ, that is

[16] D. Scolozzi, L. Cananà, *Pricing perpetual American option: an extension*, in *Annali del Dipartimento Jonico in Sistemi Giuridici ed Economicidel Mediterraneo: Società, Ambiente, Culture*, IV, 2016.

$$R_{\theta}(x) = r_{\theta}(x) - r = \mathcal{E}_g(x)(\mu - r) \qquad 0 < x < K$$

4. Conclusion. The resuls above have interesting economic implications. Optimal early exercise of options is driven by two conditions: no loss of value on exercise and rate of return equalization. Our aim, infact, is to study the expected loss rate for non optimal exercise. This involves a second condition ie the smooth past condition which is not always easy to evaluate for some pricing problems. If returns are near equalization they should exercise because this is equivalent to smooth pasting condition. Thus, even if the value function and its derivative are theoretically unknown, rates of return should be useful in determining the proximity of early exercise and to study the expected loss rate for non optimal exercise.