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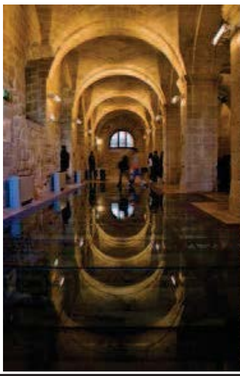

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Pricing perpetual contingent claim: an extension

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PRICING PERPETUAL AMERICAN CONTINGENT CLAIM: AN EXTENSION[*]

Abstract	
In this paper we generalise a classical result for perpetual American contingent claims. At this end we examine a more general class of payoff functions which belong to class $\mathcal{G}$. This class includes payoff function which has strictly decreasing elasticity too. Some properties of this class of payoff functions are discussed.	In questo lavoro generalizziamo un risultato noto per la valutazione di derivati perpetui che appartengono ad una classe $\mathcal{G}$. A questa classe appartengono anche funzioni payoff con elasticità strettamente decrescente.
Perpetual contingent claims - decreasing elasticity	**Derivati perpetui - Elasticità decrescente**

1. Many studies have examined nonlinear payoff functions which are not piecewise linear. This is the case, for example, of power options. See, for instance Esser [1], Heynen and Kat [2] Zhang[3] , and Tompkins[4]. See also Liu and Yao[5] for an example of S-shaped

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[*] Saggio sottoposto a referaggio secondo il sistema del doppio cieco
[1] Esser, 2003.
[2] Heynen, Kat, 1996.
[3] Zhang, 1998.
[4] Tompkins, 2000.

payoff function relating to real options. Furthermore, Blenman and Clark [6] present the class of options with constant underlying elasticity in strikes.

Perpetual american contingent claims with standard payoff function have been studied, for example, by Samuelson [7], McKean[8] , Merton [9], Karatzas[10] and Karatzas and Shreve[11], Karatzas and Shreve[12] . In this paper we examine a more general class of payoff functions, denoted by $\mathcal{G}$ which have strictly decreasing elasticity as we will see in Definition 4 .

Furthermore the following payoff functions

$$g(x) = (\max\{K - x, 0\})^{p}, \quad 0 \leq x < +\infty$$
$$g(x) = \max\{K^{p} - x^{p}, 0\}, \quad 0 \leq x + \infty,$$

with $p \in R,\ p > 0,\ K \in R,\ K > 0$ belong to class $\mathcal{G}$.

We then consider the problem of pricing a perpetual American contingent claim when the underlying asset pays no dividends; we assume the standard Black-Scholes model with constant volatility of the underlying asset return > 0 and constant risk-free interest rate r > 0.

Our main result shows that the problem to find the value of a perpetual American contingent claim which belongs to class $\mathcal{G}$ is equivalent to the related optimal stopping time problem. Moreover it is equivalent to getting the generalized solution of the free boundary problem (see Problem 1).

In this article we also provide some examples of payoff functions, such as power options, which belong to the classe $\mathcal{G}$.

The organization of the paper is as follows. In Section 2 we examine the market model and the price for a perpetual American contingent claim. In Section 3 we introduce the payoff functions defined by class $\mathcal{G}$, we study the free boundary problem associated with the pricing of a perpetual American contingent claim and state our main result. In Section 4 we provide some examples.

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[5] Liu and Yao, 1999.
[6] Blenman , Clark, 2005.
[7] Samuelson, 1965.
[8] McKean, 1965.
[9] Merton, 1973.
[10] Karatzas, 1988.
[11] Karatzas, Shreve, 1988.
[12] Karatzas, Shreve, 1991.

2. In this section, in order to deal with financial instruments as perpetual American contingent claims, we introduce the notion of financial market on $[0; +\infty[$according to Karatzas and Shreve[13] . Moreover we recall the definition of the price of the perpetual American contingent claim.

We consider the classic Black and Scholes[14] model, with a risk-free asset, called the bond, whose price $B_t$evolves according to the equation

$$dB_t = rB_t dt; \qquad B_0 = 1$$

where the rate of interest $r > 0$ is constant, and with a single risky asset, called the stock, whose price-per-share X_t follows the geometric Brownian motion

$$dX_t = \mu X_t\, dt + \sigma X_t dW_{t,}, \qquad X_0 = x > 0 \qquad (1)$$

where the volatility $\sigma > 0$ and the expected rate of return $\mu \in \mathbb{R}$ are constants.

The process $W = \{W_t\,; 0 \le t < +\infty\}$ is a standard Brownian motion on a probability space $(\Omega; \mathcal{F}; \mathrm{P})$; let us denote by $F = \{\mathcal{F}_t\}_{0\le t<+\infty}$ the filtration generated by this process, namely $\mathcal{F}_t = \sigma\,(W_s\,; 0 \le s \le t)$ augmented by all the P-null sets. We also denote by $\mathbb{E}[.]$ the expectation under the probability measure P on $(\Omega; \mathcal{F})$.

Given the function $g\colon [0 + \infty[\longrightarrow \mathbb{R}$, in this work we study perpetual American contingent claim which have payoff at time t given by

$$\Psi_t = g(\,X_t\,).$$

Definition 1. The value of the perpetual American put option at time zero is defined as

$$\mathcal{H}(x) := inf\{\gamma \ge 0;\ \exists\,(\pi, C)\ \ s.t.\ g(\,X_t\,) \le Y_t^{\gamma,\pi,C} \quad 0 \le t < +\infty \quad\} \qquad (2)$$

Where

(i) $\pi = \{\pi_t\}_{t\ge0}$ is a portfolio process, *i.e.* an F-progressively measurable process, with $\int_0^T \pi_t^2 dt < +\infty$ a.s. for all $T \in [0, +\infty[$,

(ii) $C = \{C_t\}_{t\ge0}$ is a cumulative consumption process, i.e. a nonnegative F-adapted process with increasing, right-continuous paths and with $C_0 = 0\ a.s.$,

(iii) the wealth process $Y = \{Y_t^{\gamma,\pi,C}\}_{t\ge0}$ (corresponding to initial capital γ, portfolio π and cumulative consumption C) satisfies the following stochastic differential equation:

[13]Karatzas, Shreve, 1988.
[14] Black, Scholes, 1973.

$$dY_t^{\gamma,\pi,C} = \pi_t \frac{dX_t}{X_t} + \left(Y_t^{\gamma,\pi,C} - \pi_t\right)\frac{dB_t}{B_t} - dC_t\,, \quad Y_0^{\gamma,\pi,C} = \gamma \tag{3}$$

or equivalently

$$dY_t^{\gamma,\pi,C} = \pi_t[\mu dt + \sigma dW_t] + \left(Y_t^{\gamma,\pi,C} - \pi_t\right) r\, dt - dC_t\,, \quad Y_0^{\gamma,\pi,C} = \gamma$$

We observe that π_t is the amount of the wealth $Y_t^{\gamma,\pi,C}$ that is invested in the stock at time t.

Following Karatzas and Shreve[15] the process defined by

$$Z_t = exp\left\{-\frac{\mu - r}{\sigma} W_t - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t\right\}\,, \qquad 0 \le t < +\infty$$

is a P-martingale, moreover (see for example Karatzas and Shreve[16]) there exists a unique probability measure Q on $\mathcal{F}_\infty = \sigma(W_s; 0 \le s < +\infty)$ such that:

$$Q(A) := \mathbb{E}[Z_t 1_A] \qquad \forall\, A \in \mathcal{F}_t\,, \quad 0 \le t < +\infty$$

Under this measure Q the process $\widehat{W_t}$ defined by

$$\widehat{W}_t := W_t + \frac{\mu - r}{\sigma} t\,, \qquad 0 \le t < +\infty$$

is a standard Brownian motion by the Girsanov theorem. In terms of this process we may rewrite the relation (2) as

$$dX_t = rX_t dt + \sigma X_t d\widehat{W}_t\,, \qquad X_0 = x > 0\,.$$

Lastly, we may rewrite the stochastic differential equation (3) in the form

[15] Karatzas, Shreve, 1998.

[16] Karatzas, Shreve, 1998

$$e^{-rt}Y_t^{\gamma,\pi,C} = \gamma + \sigma\int_0^t e^{-rs}\pi_s d\widehat{W}_s - \int_{]0,t]} e^{-rs}dC_s\,, \qquad t \geq 0 \tag{4}$$

From now we denote by $\widehat{\mathbb{E}}^x[\cdot]$ the expected value in $t = 0$ with respect to the probability measure Q. For details of financial markets with an infinite planning horizon (see Karatzas and Shreve[17]or Karatzas and Shreve[18]) **.**

3. In this section we introduce the setting and notation of the optimal stopping problem which is related to pricing perpetual American contingent claim and we state our main result (Theorem 1). It shows that the problem to find the value of a perpetual American contingent claim with strictly decreasing elasticity in payoff functions, is equivalent to the related optimal stopping time problem and moreover it is equivalent to getting the "generalized solution" (Definition 5) of a free boundary problem. This result extends a classical theorem for perpetual American contingent claims with standard payoff function (see Karatzas and Shreve[19]).

We first introduce the notion of elasticity for a payoff function. After that we define class $\mathcal{G}$ and discuss some properties of this class.

At this end we before introduce the definition of elasticity of g and we present the class $\mathcal{G}$, of payoff functions.

Definition 2. Let $g: [0 + \infty[\longrightarrow \mathbb{R}$ be a payoff function such that g is a decreasing continuous function on $[0 + \infty[$ with $g(0) > 0$ and $g(+\infty) = 0$.

Let $K := sup\{x \geq 0;\ g(x) > 0\}$ and let us suppose that g is differentiable on the interval $[0K[$.

The elasticity of g is defined as:

$$\mathcal{E}_g(x) := \frac{xg'(x)}{g(x)} = x\frac{d}{dx}\log\big(g(x)\big)\,, \qquad x \in\,]0, K[\,.$$

Definition 3. Let $l := inf_{0<x<K}\mathcal{E}_g(x)$ and $m := inf\left\{x > 0\,;\ \mathcal{E}_g(x) + \frac{2r}{\sigma^2} \leq 0\right\}$ with the usual convention that: $inf\emptyset = +\infty$.

In the following proposition we give some properties of the elasticity of payoff function g.

Proposition 1: Let $g: [0 + \infty[\longrightarrow \mathbb{R}$ be a payoff function such that g is a decreasing continuous function on $[0 + \infty[$ with $g(0) > 0$ and $g(+\infty) = 0$.

[17] Karatzas, Shreve, 1998.
[18] Karatzas, Shreve, 1991.
[19] Kartatzas, Shreve, 1998.

Let $K := sup\{x \geq 0;\ g(x) > 0\}$ and let us suppose that g is continuously differentiable on the interval $]0, K[$.
Then

(i) $sup_{0<x<K}\ \mathcal{E}_g(x) = 0$;
(ii) $\lim_{x\to+\infty} x^\alpha\, g(x) = 0$ for all $\alpha > 0 \Rightarrow\ l = inf_{0<x<K}\mathcal{E}_g(x) = -\infty$;
(iii) $m = inf\left\{x > 0\ ;\ \mathcal{E}_g(x) + \frac{2r}{\sigma^2} \leq 0\right\} > 0$ (with the usual convention that $inf\emptyset = +\infty$);
(iv) Let $g \in C^2(]0, K[)$ we have:

$$\frac{1}{2}\sigma^2 x^2 g''(x) + r\, xg'(x) - r\, g(x) = \frac{\sigma^2 g(x)}{2}\left\{x\mathcal{E}_g'(x) - \left(1 - \mathcal{E}_g(x)\right)\left(\mathcal{E}_g(x) + \frac{2r}{\sigma^2}\right)\right\}$$
$x \in]0, K[$;

(v) If g has decreasing elasticity on]0,m[we have

$$\frac{1}{2}\sigma^2 x^2 g''(x) + r\, xg'(x) - r\, g(x) \leq 0;$$

(vi) $\lim_{x\to 0^+} x\,\mathcal{E}_g'(x) = 0\ \Rightarrow \exists\bar{x} > 0 : \forall x \in]0, \bar{x}[$

$$\frac{1}{2}\sigma^2 x^2 g''(x) + r\, xg'(x) - r\, g(x) < 0.$$

Proof:

We suppose that condition (i) is false, then we have $\mathcal{E}_g\ (x) < a\ (a > 0)\ x \in]0, K[$; we easily obtain $x^a g(x) \leq u^a g(u)$ when $u \to 0$, so we have a contradiction. Thus condition (i) is true.

In the same way we prove condition (ii).

We use condition (i) to prove (iii).

Lastly, condition (iv), (v) and (vi) are clear.

Remark 1: The payoff of power option *(*see section *4)* satisfies $\lim_{x\to 0^+} x\,\mathcal{E}_g'(x) = 0$.
We now introduce class $\mathcal{G}$ of payoff functions.

Definition 4 We say that the payoff function $g : [0, +\infty[\to \mathbb{R}$ belongs to class $\mathcal{G}$ if the following conditions are satisfied:
(g1) g is a decreasing continuous function on $[0, +\infty[$ wit*h* $g(0) > 0$ *a*nd $g(+\infty) = 0$;
(g2) $g \in C^2(]0, K[)$ where $K = sup\{x \geq 0;\ g(x) > 0\}$;
(g3) the elasticity $\mathcal{E}_g(x)$ *of* g satisfies the following property:

$$x\mathcal{E}_g'(x) \leq \left(1 - \mathcal{E}_g(x)\right)\left(\mathcal{E}_g(x) + \frac{2r}{\sigma^2}\right), \qquad x \in]0, m[.$$

A function g which satisfies properties $g_1$and g_2 and which has decreasing elasticity function on $]0, K[$ belongs to class $\mathcal{G}$.

We now formulate the free boundary problem associated with the pricing of a perpetual American option as follows.

Problem. Let $g \in \mathcal{G}$ and let $K = sup\{x \geq 0;\ g(x) > 0\}$. Find $b \in]0, K[$ and a function v in the space $C^0([0, +\infty[) \cap C^1(]0, +\infty[) \cap C^2(]0, +\infty[\backslash\{b\})$ such that

$$\begin{cases} \frac{1}{2}\sigma^2 x^2 v''(x) + rxv'(x) - rv(x) = 0, & b < x < +\infty \\ v(+\infty) = 0 & \\ v(x) = g(x), & 0 \leq x \leq b \end{cases}$$

The following proposition gives the solution (b; v) of Problem 1. In the following one we call value function the solution v. Note that $b \in \mathbb{R}$ is called the critical stock price.

Proposition 2 Let $g \in \mathcal{G}$ and let $K = sup\{x \geq 0;\ g(x) > 0\}$.
Let $l = inf_{0<x<K}\mathcal{E}_g(x)$ *and assume that* $l < -\frac{2r}{\sigma^2}$.

Let $b = m$ and define

$$v(x) = \begin{cases} g(x) & 0 \leq x \leq b \\ g(b)\left(\frac{b}{x}\right)^{\frac{2r}{\sigma^2}} & b < x < +\infty \end{cases}$$

Then (b, v) is the unique solution of Problem 1.

Proposition 3 *(*Asymptotic version*)*
Let us consider a payoff function $g \in \mathcal{G}$. Assume that $K = +\infty$ and $l \geq -\frac{2r}{\sigma^2}$.
We can find a sequence of payoff functions $(g_n)_{n\in\mathbb{N}}$ such that:

(i) $sup\{x > 0\ ;\ g_n(x) > 0\} < +\infty$
(ii) $\lim_{n\to+\infty} g_n(x) = g(x)$ *for all* $x \geq 0$,
(iii) let (b_n, v_n) be the unique solution of Problem 1 for the payoff function g_n (given in Proposition 2), it results that
$\lim_{n\to+\infty} b_n(x) = +\infty$ *and* $\lim_{n\to+\infty} g_n(x) = g(x)$ *for all* $x \geq 0$.

Definition 5 Let $g \in \mathcal{G}$. Let $K = sup\{x \geq 0;\, g(x) > 0\}$ and let $l = inf_{0<x<K} \mathcal{E}_g(x)$.

We say that *(b,v)* is the generalized solution if the following condition is satisfied*: if* $l < -\frac{2r}{\sigma^2}$ *then* (b, v) is the solution of Problem 1 given in Proposition 2, otherwise $b = +\infty$ *and* $v = g$.

Before stating our main result, we recall the optimal stopping time problem.

Definition 6 We define

$$V(x) \coloneqq \sup_{\tau \in S} \widehat{\mathbb{E}}^x \left[e^{-r\tau} g(X_\tau)\right]$$

where S is the class of all $\mathcal{F}$-stopping times.

The optimal stopping time problem is to find the stopping time $\tau_* \in S$ such that

$$V(x) := \widehat{\mathbb{E}}^x\left[e^{-r\tau_*} g(X_{\tau_*})\right].$$

We now state the main result of this paper. It is a generalization of a well-known result for the vanilla option (see Karatzas and Shreve [20]).

Theorem 1 Let $g \in \mathcal{G}$ and let (b, v) be the generalized solution introduced in Definition 5. Let $H(x)$ be the price as defined in (2) and let $V(x)$ be the function defined in (5).

Then we have

$$\forall x > 0 \;\; v(x) = H(x) = V(x) = \sup_{\tau \in S} \widehat{\mathbb{E}}^x \left[e^{-r\tau} g(X_\tau)\right]. \qquad (6)$$

The optimal stopping time τ_b is given by $\tau_b \coloneqq inf\,\{t \geq 0;\, X_t \leq b\}$ with the usual convention that $\inf \emptyset := +\infty$*;* so we obtain

$$\forall\, x > 0 \quad v(x) = H(x) = V(x) = \widehat{\mathbb{E}}^x \left[e^{-r\tau_b} g(X_{\tau_b})\right].$$

Theorem 1 is a direct consequence of Theorem 2 and Theorem 3, which will be stated and proved later.

Remark 2 Note that in the case $l \geq -\frac{2r}{\sigma^2}$ we have $b = +\infty$ and $v = g$. Then $\tau_b = 0$ and, taking into account that $X_0 = x$, we get

$$\widehat{\mathbb{E}}^x\left[e^{-r\tau_b} g(X_{\tau_b})\right] = g(x) = v(x).$$

[20] Karatzas, Shreve, 1998.

We examine, now, the relationship between the value function and the optimal stopping time problem.

Proposition 4 Let $g \in \mathcal{G}$. Let (b, v) be the generalized solution introduced in Definition 5 and let

$$\tau_b := inf \{t \geq 0;\ X_t \leq b\}$$

Then

$\forall\, x > 0 \quad v(x) = \widehat{\mathbb{E}}^x\left[e^{-r\tau_b} \quad g\,(X_{\tau_b}\,)\right].$

Therefore

$$\forall x > 0 \qquad v(x) \leq V\,(x)$$

where V is the function introduced in Definition 6.

Theorem 2 Let $g \in \mathcal{G}$ and let (b, v) be the generalized solution introduced in *Definition 5.*

Then

$$\forall x > 0 \qquad v(x) = V\,(x) = \sup_{\tau \in S} \widehat{\mathbb{E}}^x\,[e^{-r\tau}\, g(X_\tau)].$$

Moreover, denoting by

$$\tau_b := inf \{t \geq 0;\ X_t \leq b\}$$

we have

$$\forall x > 0 \qquad v(x) = V\,(x) = \widehat{\mathbb{E}}^x\left[e^{-r\tau_b} \quad g\,(X_{\tau_b}\,)\right].$$

Proof:

By Proposition 4 we have only to prove that $v(x) \geq V\,(x)$ for all $x \geq 0$.

We know that, for all $\tau \in S$ and for all $n \in \mathbb{N}$:

$$\widehat{\mathbb{E}}^x\left[e^{-r(\tau\wedge n)}\ v\,(X_{\tau\wedge n}\,)\right] - v(x) = \widehat{\mathbb{E}}^x\left[\int_0^{\tau\wedge n} e^{-rt}\left(\frac{\sigma^2}{2}X_t^2 v''(X_t) + r\,X_t v'(X_t) - r\,v(X_t)\right)dt\right]. \quad (7)$$

From (7) and knowing that the value function v of class

$C^0([0,+\infty[) \cap C^1(]0,+\infty[) \cap C^2(]0,+\infty[\backslash\{b\})$ satisfies the following property

$$\frac{\sigma^2}{2}x^2 v''(x) + r\,xv'(x) - rv\,(x)\begin{cases} < 0 & 0 < x < b \\ = 0 & b < x < +\infty \end{cases} \qquad (8)$$

and taking into account that $v \geq g$, we obtain

$$v(x) \geq \widehat{\mathbb{E}}^x\left[e^{-r(\tau \wedge n)}\ v\left(X_{\tau \wedge n}\right)\right] \geq \widehat{\mathbb{E}}^x\left[e^{-r(\tau \wedge n)}\ g\left(X_{\tau \wedge n}\right)\right].$$

By considering the limit of this expression when $n \to +\infty$ and applying the dominated convergence theorem (taking into account that
$e^{-r\,(\tau \wedge n)} g(X_{\tau \wedge n}) \leq g(0)$) we obtain

$$v(x) \geq \widehat{\mathbb{E}}^x\left[e^{-r\tau}\ g\left(X_\tau\right)\right] \qquad for\ all \qquad \tau \in S.$$

Then we have $v(x) \geq V(x)$ and the proof is complete.

The following proposition deals with the relationship that exists between the generalized solution introduced in Definition 5 and the price at time zero of the perpetual contingent claim introduced in Definition 1.

Lemma 1. Let $g \in \mathcal{G}$ and let (b, v) be the generalized solution introduced in Definition 5.

Then

$$\forall x > 0 \quad v(x) \leq H(x).$$

Proof: The cases $b = +\infty$ and $v = g$ are easy to prove.
We now assume that $b \in \mathbb{R}$. Let $\gamma > 0$ and let us assume that there exist a portfolio process $\pi = \{\pi_t\}_{t \geq 0}$ and a cumulative consumption process $C = \{C_t\}_{t \geq 0}$ such that

$$g(X_t) \leq Y_t^{\gamma,\pi,C}, \qquad t \geq 0$$

or, equivalently, by choosing: $t = \tau_b \wedge n$ we have

$$e^{-r(\tau_b \wedge n)}\ g\left(X_{\tau_b \wedge n}\right) \leq \gamma + \sigma \int_0^{\tau_b \wedge n} e^{-rs}\, \pi_s d\widehat{W}_s\ .$$

By a standard argument we have:

$$\widehat{\mathbb{E}}^x\left[e^{-r(\tau_b \wedge n)}\ g\left(X_{\tau_b \wedge n}\right)\right] \leq \gamma\ ..$$

By considering the limit of this expression when $n \to +\infty$ and applying the dominated convergence theorem we obtain

$$v(x) = \widehat{\mathbb{E}}^x\left[e^{-r\tau_b}\ g\left(X_{\tau_b}\right)\right] \leq \gamma.$$

Lemma 2 Let $g \in \mathcal{G}$ and let (b, v) be the generalized solution introduced in Definition 5.

Then

$$\forall x > 0 \qquad v(x) \geq H(x).$$

Proof:

By applying Itoˆ's lemma we have

$$e^{-rt}v(X_t) - v(x) = \int_0^t e^{-rs}\left(\frac{\sigma^2}{2}X_s^2 v''(X_s) + r\,X_s v'(X_s) - r\,v(X_s)\right)ds + \sigma\int_0^t e^{-rs}\,X_s v'(X_s)d\widehat{W}_s$$

e^{-rt} and thus, from (8), we obtain

$$e^{-rt}v(X_t) - v(x) \leq \sigma\int_0^t e^{-rs}\,X_s v'(X_s)d\widehat{W}_s\,.$$

Taking into account that $v \geq g$ we have for $0 \leq t < +\infty$

$$e^{-rt}g(X_t) \leq \quad e^{-rt}v(X_t) \leq v(x) + \sigma\int_0^t e^{-rs}\,X_s v'(X_s)d\widehat{W}_s\,.$$

By considering the portfolio process π defined by $\pi_t := X_t\,v'(X_t)$ and the consumption process C defined by $C_t \equiv 0$, from Definition 1 and by using (4) we obtain $v(x) \geq H(x)$.

From Lemma 1 and Lemma 2 we obtain the following theorem.

Theorem 3 Let $g \in \mathcal{G}$ and let (b, v) be the generalized solution introduced in Definition 5. Then

$$\forall x > 0 \qquad v(x) = H(x).$$

In the next remark we give a portfolio/consumption process pair $(\bar{\pi}, \bar{C})$ such that the corresponding wealth process matches the claim's price process.

Remark 3. Let (b, v) be the solution of Problem 1. In analogy with Karatzas and Shreve[21] and Karatzas[22] let us consider*:*

the wealth process $\bar{Y} = \left\{Y_t^{\bar{y},\bar{\pi},\bar{C}}\right\}_{t\geq 0}$ corresponding to initial capital $\bar{y} := v(x)$, hedging portfolio

$$\bar{\pi}_t := X_t\,v'(X_t)$$

[21] Karatzas, Shreve, 1998.

[22] Karatzas, 1996

and cumulative consumption

$$\bar{C}_t := -\int_0^t \left(\frac{\sigma^2}{2} X_s^2 v''(X_s) + r\, X_s v'(X_s) - r\, v(X_s)\right) ds$$
$$= -\int_0^t \left(\frac{\sigma^2}{2} X_s^2 g''(X_s) + r\, X_s g'(X_s) - r\, g(X_s)\right) 1_{]0,b[}(X_s) ds$$

From (4) we get
$$Y_t^{\bar{\gamma},\bar{\pi},\bar{C}} = v(X_t), \qquad t \geq 0.$$

4.We give some examples of functions which belong to class $\mathcal{G}$.

Example 1 Let $K \in \mathbb{R}, K > 0$. The following payoff functions $g : [0,+\infty[\to \mathbb{R}$ belong to class $\mathcal{G}$ and they have $inf \mathcal{E}_g(x) = -\infty$:

(a) powered option:
$$g(x) = (\max\{K - x, 0\})^p \qquad (p \in \mathbb{R}, p > 0);$$
(b) power option:
$$g(x) = max\{K^p - x^p, 0\} \qquad (p \in \mathbb{R}, p > 0);$$

(c) general power option (or polynomial option):

$$g(x) = max\left\{\sum_{i=1}^{n} a_i\left((2K - x)^i - K^i\right), 0\right\} \qquad (a_i > 0, n \in \mathbb{N});$$

(d) parabola option:

$$g(x) = max\left\{a\left((2K - x)^2 + b\left((2K - x) - K\right)\right), 0\right\} \qquad (a > 0, b \geq 0).$$

(e) *S*-shaped payoff function

$$g(x) = \begin{cases} K - K\sin^2\left(\frac{\pi}{2K}x\right) & 0 \leq x \leq K \\ 0 & K \leq x < +\infty \end{cases}$$

Lastly, we give some example of payoff function which belongs to class $\mathcal{G}$, $K = +\infty$ and with decreasing elasticity function.

Example 2 The function

$$g(x) = e^{-x}, \qquad x \geq 0$$

where $\inf \mathcal{E}_g(x) = -\infty$.

Example 3 The function

$$g(x) = \frac{1}{1+x}, \qquad x \geq 0$$

where $\inf \mathcal{E}_g(x) = -1$.

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