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Abstract

We model an agency relationship in which the agent's cost is non-monotonic with respect to type and the type is correlated with a public *ex-post* signal. The principal can use lotteries to exploit the type-signal correlation within the limit of the agent's liability. We establish conditions for first-best implementation, highlighting two effects on contractual design. First, the structure of the optimal lottery varies across types and, for each type, it depends on whether the cost is U shaped or reverse U shaped with respect to type. Second, as compared to the case of monotonic cost, the design of incentive compatible lotteries is easier when the cost is U shaped, more difficult when the cost is reverse U shaped. The root of the second effect is that incentives are non-monotonic either below or above some interior types. The two effects involve that non-monotonicity is unfavorable to the principal when the cost is reverse U shaped. This conclusion is at odds with the wisdom, concerning settings without correlated information, that non-monotonicity, which triggers countervailing incentives, enhances contracting.

Keywords: Non-monotonic cost; countervailing incentives; correlated information; limited liability; first-best implementation

J.E.L. Classification Numbers: D82

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1 Introduction

We investigate the optimal contractual design in agency relationships in which the cost incurred by the agent in the trade with the principal is non-monotonic with respect to his private information (the type). Whereas agency problems of this kind have already been studied, the novelty of our analysis is that it focuses on environments in which a publicly observable signal conveys information about the type after the contractual transaction has occurred, and the agent is protected by limited liability.

Characterizing the optimal contract in agency problems with non-monotonic cost, Maggi and Rodriguez-Clare [16] (MRC hereafter) show that the profit targeted to each possible type of agent depends finely on the shape of the cost with respect to the type. The reason is that the agent has countervailing incentives to misrepresent information. That is, the agent is tempted to either overstate or understate his type, depending on whether the type belongs to the increasing or the decreasing side of the cost. Therefore, for each given type, the contractual solution depends on the exact side of the cost function to which that type belongs. This, in turn, depends on whether the cost is U shaped or reverse U shaped. This result is rather general and, not surprisingly, it also emerges in other studies on agency problems with countervailing incentives.[1]

Principal-agent relationships with *ex-post* informative signals are examined in another category of contract-theoretic models, pioneered by Riordan and Sappington [19].[2] In those models, each type of agent is faced with a lottery of profits, in addition to being assigned a cost reimbursement. Incentive constraints depend on the properties of the conditional probabilities of the signal realizations and on the properties of the cost function. The former determine the effectiveness of the lottery at extracting the gain in cost reimbursement possibly associated with a fake report; the latter determine the magnitude of that gain. Riordan and Sappington [19] show that the characteristics of the cost are irrelevant, and first best is implemented, as long as the vectors of conditional probabilities of the signals are linearly independent across

[1] Agengy problems with countervailing incentives to misrepresent information are found in: procurement, when firms are specialized (Boone and Schottmuller [4]) and when they have a privileged knowledge of the quality of a public signal about their production costs (Che and Sappington [5]); regulation, when the firm incurs a fixed cost inversely related with the privately known marginal cost (Lewis and Sappington [14], MRC), with non-linear pricing under price cap (Jullien [11]), when two-product monopolists face complementary demands (Aguirre and Beitia [2]) and when utilities are subject to universal service obligations (Poudou *et al.* [18]); labour and financial contracts, when the hidden effort exerted by the agent is complementary to his privately known ability in accomplishing the task for the principal (Ollier and Thomas [17]); vertical relationships, when retailers need to specialize some assets before contracting with the upstream suppliers (Acconcia *et al.* [1]); conflicts on investment levels between uninformed shareholders and informed managers (Degryse and de Jong [8]); landowner-farmer contracts with up-front capital endowments (Lewis and Sappington [15]); government-taxpayer relationships, when the government wishes to improve the wellbeing of low-skill individuals by taxing high-skill individuals but is aware that the latter will emigrate if the utility they attain within the country is less than the utility they would attain in other jurisdictions (Krause [13]).

[2] For instance, in procurement and regulation, information about the productivity or the production cost of the firm is obtained by observing the behaviour or the market performance of another firm operating in the same sector in a neighboring economy. Information is also conveyed by an audit of the firm's activity or an *ex-post* evaluation of the firm's performance. In these and many other instances, the newly acquired information is observable and verifiable, and the principal can use it in the contractual offer to the agent.

types.[3] On the other hand, when this condition is violated the principal needs to design a lottery which is sufficiently unfavorable to types that gain in cost reimbursement, if they make a fake report. That lottery is found to be such that the agent is rewarded only for the signal that is least likely to be drawn by types that would gain in cost reimbursement with a fake report. With the cost being increasing in type, these are low types exaggerating information, and the lottery is effective as long as the cost is either convex or not highly concave. Intuitively, in those cases, the gains in cost reimbursement for low types exaggerating information are not too high relative to the losses in cost reimbursement for high types understating information. Hence, the lottery will permit the extraction of the gains in cost reimbursement from low types without being attractive to high types.

Considering limited liability on the agent's side, which seems to be the rule rather than an exception in practice, Gary-Bobo and Spiegel [10] further demonstrate that the principal is most likely to implement first best if she compensates the agent exactly as shown by Riordan and Sappington [19]. However, in that study attention is restricted to the case of a convex cost. In a companion paper (Danau and Vinella [7]), we highlight that an efficient outcome may be at reach under limited liability even when the concavity of the cost function is more pronounced than is admitted by the lottery proposed by Riordan and Sappington [19]. Effecting that outcome requires using a different lottery, which is yet feasible only if the agent's liability is not too tight. Noticeably, in all these models the cost is taken to be monotonic with respect to type, an assumption which we relax in this study. This permits us to investigate what effects are induced on the design of the optimal contract when the cost of the agent is U shaped and when it is reverse U shaped with respect to type, in line with MRC.

To consider alternative shapes of the cost function, we assume, similarly to MRC, that the agent's total cost includes a variable cost of production, which is linear in type, and a fixed cost of production, to be interpreted as an opportunity cost of being in the trade with the principal, which declines with the type and can take any shape. Moreover, in line with the literature on contractual design with informative signals, we allow for a signal correlated with the type to be publicly observed *ex post* and investigate under what conditions the first-best allocation is effected. Following Demougin and Garvie [9], Gary-Bobo and Spiegel [10] and Danau and Vinella [7], we assume that the agent's profit cannot fall below a certain value regardless of the signal realization. Under limited liability, the optimal contract will depend finely on the shape of the cost.[4]

We show that non-monotonicity induces two specific effects on the lottery design. To explain them, it is useful to recall how the lotteries should be structured to attain contractual efficiency when the cost increases monotonically with the type, as developed extensively in Danau and

[3] At the first-best allocation, the entire surplus is retained from the agent and the volume of trade is efficient. The result that this allocation is effected contractually under the assumption of linear independence is also found by Crémer and McLean [6] in an auction context with correlated private information across participants.

[4] Remarkably, one particular case would be that in which the agent is not protected by limited liability and, yet, whether or not full efficiency is attained depends on the shape of the cost, as shown by Riordan and Sappington [19]. Indeed, from Gary-Bobo and Spiegel [10] we know that the lottery that Riordan and Sappington [19] find to attain full efficiency as long as the cost is not highly concave, is the locally incentive compatible lottery under which the limited liability constraints are relaxed to the utmost.

Vinella [7]. In that case, for any intermediate type θ, lower (more efficient) types gain in cost reimbursement, if they announce θ; higher (less efficient) types lose in cost reimbursement, instead. It follows that the lottery designed for type θ should accomplish two tasks to achieve efficiency. First, it should extract any potential gain in cost reimbursement from types below θ; second, it should not grant any benefits which overcome the potential loss in cost reimbursement to types above θ. In the setting here considered, the signals are obviously used in lotteries to pursue analogous purposes for any intermediate type. Nonetheless, the following two specific effects will be identified.

The *first effect* is that the structure of the optimal lottery varies across types and, for each type, it depends on whether the total cost is U shaped or reverse U shaped. To see why, consider that when the cost is U shaped low types gain in cost reimbursement if they understate information, whereas high types gain if they overstate information. The converse occurs when the total cost is reverse U shaped, instead. For any type θ in the interior of the feasible set, a proper choice of the lottery structure cannot be made without considering whether lower or higher neighboring types gain in cost reimbursement, if they report θ. The relevance, for contractual design, of the specific type realization and the shape of the total cost is not novel with respect to the findings of MRC. However, whereas in MRC those aspects determine what exact incentives must be accounted for in contractual design, in our framework they dictate how the lottery of any given type should be structured to be incentive compatible for each and every other type.

The *second effect* of non-monotonicity is that, as compared to the case of monotonic cost, the design of incentive compatible lotteries is easier when the cost is U shaped but more difficult when the cost is reverse U shaped. This effect arises because, for any given type θ, some types which lie on one side of θ gain in cost reimbursements, whereas the other types on the same side lose, if they report θ. First consider a U shaped cost. Very low types may be penalized in cost reimbursement, if they mimic some θ on the increasing side of the cost. Similarly, very high types may be penalized, if they mimic some θ on the decreasing side of the cost. This facilitates the task of designing lotteries such that the intermediate types are not attractive reports to both lower and higher types. For instance, if θ belongs to the decreasing side of the cost, then not only lower types but also some of the higher types lose in cost reimbursement, if they mimic θ. Hence, incentive compatibility is more easily attained than with a monotonic cost. Importantly, with a reverse U shaped cost, the second effect acts in the opposite direction, making it more difficult to reconcile the incentives of the types below and above θ. For instance, if θ belongs to the increasing side of the cost, then not only lower types but also some of the higher types may gain in cost reimbursement, if they report θ. This occurs, indeed, when the cost is sharp sloping, in which case it is impossible to make the lottery of type θ incentive compatible for all the other types.

Taken together, the two effects of non-monotonicity explain the following *general result* of our study. For any given level of the agent's liability, under some regularity conditions, the principal is able to attain the first-best outcome by means of a suitable choice of the lotteries, if and only if either the total cost is U shaped, or it is reverse U shaped and not too sharp sloping. This points to the conclusion that, in agency relationships with correlated information, facing an agent with reverse U shaped cost makes the screening task more problematic for the principal than facing an agent with monotonic cost, at odds with the wisdom proper of the literature on agency problems with countervailing incentives.

To the same conclusion points another, more *specific result*, which we derive through a comparison of the case in which the cost is sharp sloping reverse U shaped with the case in which the cost increases monotonically with type. This latter case arises when the declining effect induced by the opportunity cost is not pronounced enough to countervail the rising effect induced by the production cost. The result is that, in the non-monotonic setting, first-best is implemented in the presence of lower degrees of concavity of the opportunity cost, given an equal level of the agent's liability. Therefore, non-monotonicity of the cost restricts the family of cost functions for which the principal attains full efficiency.

This paper is firstly related to the studies on agency problems with countervailing incentives to misrepresent information. Whereas in the pioneering study of Lewis and Sappington [14] attention is restricted to the case of a reverse U shaped cost, Maggi and Rodriguez-Clare [16] and various authors thereafter find that the contractual solution changes as the cost takes different shapes.[5] It is thus not surprising that the shape of the cost will play an important role also in our analysis. Peculiar to our framework is that the shape of the cost affects the choice of a lottery, which cannot be the case in settings without informative signals.

Our paper is also related to the literature on contractual design with *ex-post* informative signals and limited liability on the agent's side. Within that literature, in line with Demougin and Garvie [9], Gary-Bobo and Spiegel [10], and Danau and Vinella [7], we take limited liability to be represented as a lower bound on the profits that can be assigned to the agent in the various possible contingencies.[6] This formalization admits a natural interpretation: it reflects a commitment of the principal to preserve the agent's financial viability, along a widespread practice. In addition, it enables us to focus on the restrictions that are imposed by the agent's limited liability on the optimal lottery design, rather than on the trade-off between optimal transfers and trade distortions.[7]

[5] Jullien [11] extends the analysis of MRC by relaxing the assumption of full participation and exploring common values situations in which the private information parameter has a direct impact on the principal's welfare.

[6] Unlike in the other two studies cited in the text and in the model here developed, in Gary-Bobo and Spiegel [10] the *ex-post* signal is an exogenous shock which affects the cost of production. Because of this, in their model not only the compensation but also the allocation depends on the signal realization. In this respect, their study comes closer to those about contractual design with correlated information, such as the study of Crémer and McLean [6] who consider an auction context with correlated private information across participants. However, no substantial difference follows in terms of the principal's achievements.

[7] A different form of limited liability, mirroring the principal's imperfect ability to tax the agent, would require the transfer to the agent not being too low in any possible state. That case is explored by Demougin and Garvie [9] and Kessler *et al.* [12] in models without non-monotonicity.

1.1 Outline

The reminder of the article is organized as follows. In section 2 we describe the model, we state the principal's programme and we characterize the first-best allocation. Section 3 offers an overview of our analytical approach and main findings. The analysis of the lottery design is developed in section 4. In section 5 we provide conditions under which first best is attained for different shapes of the agent's opportunity cost. In section 6 we discuss the benefit that non-monotonicity grants to the principal in settings with correlated information, relative to settings without correlated information, as considered by MRC. Section 7 briefly concludes. Most mathematical details are relegated to an appendix.

2 The model

A principal contracts with an agent for the provision of q units of some good (or service). They are both risk neutral.

Consumption of q units of the good yields a gross utility of $S(q)$. We assume that the function $S(\cdot)$ is twice continuously differentiable and such that $S'(\cdot) > 0$ and $S''(\cdot) < 0$. Furthermore, $S(0) = 0$ and the Inada's conditions are satisfied.

To be in the trade with the principal and supply q units of the good, the agent incurs a total cost of $C(q, \theta) = \theta q + K(\theta)$. The marginal cost $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$, where $\overline{\theta} > \underline{\theta} > 0$, captures the agent's *efficiency* in the production activity. $K(\theta)$ represents the agent's *opportunity cost* of renouncing to other businesses, which depends on efficiency. Assuming that $K(\cdot)$ is twice continuously differentiable, we assume that $K'(\theta) < 0$ for all θ : the less efficient that the agent is in the relationship with the principal, the worse that his outside opportunity is. We also take $K''(\cdot)$ to have a constant sign across types, which will be functional to the exposition of results below.

Information structure Nature draws θ and the agent observes its realization (his type) before receiving the contractual offer. The public beliefs about θ are reflected in the continuously differentiable density function $f(\theta)$. The associated cumulative distribution function is denoted $F(\theta)$. The agent's marginal cost is correlated with a random signal s (the "state" of nature). This is hard information and can be included in a legally enforceable contract. For instance, in regulatory settings the signal can be the behavior or the market performance of another firm, which conveys information about production costs. It can also be the outcome of an audit or performance evaluation. We assume that the signal is drawn from the discrete support $N \equiv \{1, .., n\}$, where $n \geq 3$, and publicly observed after the contract has been signed and the level of output has been chosen (or the output has been delivered). The degree of correlation between type and signal is commonly known prior to the contractual offer being made. It is measured by the probability $p_s(\theta) > 0$ of observing signal $s \in N$ conditional on the type being θ. The function $p_s(\cdot)$ is twice continuously differentiable for all types.

The contract The Revelation Principle applies and the principal can restrict attention to direct mechanisms in which the agent reports truthfully (or, equivalently, he picks the contractual option targeted to his type within the menu offered by the principal). As the signal is publicly observed *ex post*, it can be used to condition the compensation to the agent. For instance, when a regulator (or public procurer) audits the activity of the regulated firm (or contractor), the compensation to the firm can be made contingent not only on firm's report (or contractual choice) but also on the outcome of the audit, which is informative about the firm's efficiency. Formally, the take-it-or-leave-it offer is a profile of allocations $\{q(\theta), \mathbf{t}(\theta)\}$, $\forall\theta$, where $q(\theta)$ is the quantity an agent of type θ will produce and $\mathbf{t}(\theta) \equiv (t_1(\theta), ..., t_n(\theta))$ is the vector of the monetary transfers he will receive in states 1 to n. Considering both the production cost and the opportunity cost, the profit that type θ' obtains when θ is announced and signal s is realized, is given by $\widetilde{\pi}_s(\theta|\theta') \equiv t_s(\theta) - (\theta' q(\theta) + K(\theta'))$. We let $\pi_s(\theta) \equiv \widetilde{\pi}_s(\theta|\theta)$ and denote the lottery of profits designed for an agent of type θ as the vector $\boldsymbol{\pi}(\theta) \equiv (\pi_1(\theta), ..., \pi_n(\theta))$. As usual, it is more convenient to consider the profits (rather than the transfers) as the decision variables. The principal rewards the agent in state s if $\pi_s(\cdot) > 0$ and punishes him if $\pi_s(\cdot) < 0$. Rewriting $\widetilde{\pi}_s(\theta|\theta')$ as

$$\widetilde{\pi}_s(\theta|\theta') = \pi_s(\theta) + (\theta - \theta')\, q(\theta) + K(\theta) - K(\theta'),$$

it becomes visible that the payoff of type θ' includes two components. The first is the profit this type receives when it reports θ and signal s is realized. The second is the cost reimbursement type θ' obtains, net of the cost it incurs to produce the $q(\theta)$ units of the good recommended by the principal from type θ. Therefore, the expected payoff of an agent of type θ' who announces θ is given by

$$\mathbb{E}_s[\widetilde{\pi}_s(\theta|\theta')] = \sum_{s\in N} p_s(\theta')\, \pi_s(\theta) + (\theta - \theta')\, q(\theta) + K(\theta) - K(\theta'), \tag{1}$$

and includes the expected value of the *lottery* of profits he is faced with and the net *cost reimbursement.*

The principal's programme The principal's programme is formulated as follows:

$$\underset{\{q(\theta);\boldsymbol{\pi}(\theta),\forall\theta\}}{Max} \int_{\underline{\theta}}^{\overline{\theta}} \{S(q(\theta)) - (\theta q(\theta) + K(\theta)) - \mathbb{E}_s[\pi_s(\theta)]\}\, dF(\theta)$$

subject to

$$\mathbb{E}_s[\pi_s(\theta)] \geq \mathbb{E}_s[\widetilde{\pi}_s(\theta'|\theta)],\ \forall\theta,\theta', \tag*{$(IC_\theta^{\theta'})$}$$

$$\mathbb{E}_s[\pi_s(\theta)] \geq 0,\ \forall\theta, \tag*{(PC_θ)}$$

$$\pi_s(\theta) \geq -L,\ \forall\theta,\ \forall s \in N. \tag*{(LL_θ^s)}$$

$(IC_\theta^{\theta'})$ is the incentive constraint whereby an agent of type θ has no incentive to report $\theta' \neq \theta$ (or to pick the contractual option designed for type θ'). (PC_θ) is the *ex-ante* participation constraint which ensures that type θ obtains a non-negative profit in expectation. (LL_θ^s) is the limited liability constraint which ensures that the highest deficit the agent is exposed to does not exceed $L > 0$ in any state s. The formulation of this constraint mirrors the principal's commitment to prevent the agent from becoming so financially distressed that the activity must be interrupted, at least as long as the agent does not attempt to conceal information.[8]

The first-best allocation At the first-best allocation the quantity is such that $S'(q^*(\theta)) = \theta$, $\forall\theta$. Given the properties of the function $S(\cdot)$, the first-best quantity is positive and unique for any given value of θ, and the function $q^*(\cdot)$ is continuous for all values of θ. Moreover, the profits are such that all surplus is extracted from any type of agent, namely:

$$\mathbb{E}_s[\pi_s(\theta)] = \sum_{s\in N} p_s(\theta)\,\pi_s(\theta) = 0, \ \forall\theta. \tag{2}$$

We will look for conditions under which the principal decentralizes this allocation through the contract.

The properties of the conditional probabilities In the framework we consider, the conditional probabilities of the signals display the following properties:

$$\frac{p_1(\theta)}{p_1(\theta')} > \frac{p_s(\theta)}{p_s(\theta')} > \frac{p_n(\theta)}{p_n(\theta')}, \ \forall\theta,\theta' \text{ such that } \theta\geq\theta', \ \forall s\neq 1,n \tag{3}$$

$$p_1''(\theta) < 0 \ \text{ and } \ p_n''(\theta) < 0, \ \forall\theta. \tag{4}$$

The conditions in (3) represent a weaker version of the monotonic likelihood property, which is standard in the contract-design literature. Here, it is required to hold for any triplet of signals that includes the two extreme signals 1 and n; it does not need to hold for all possible signals. Concavity of the conditional probability of some signals, as here imposed by (4) on 1 and n, is also required in Riordan and Sappington [19] (Corollary 1.4) and Gary-Bobo and Spiegel [10] (Assumption 1). It is thus not surprising that it will have bite in our model as well. Taken together, (3) and (4) entail some monotonicity on the expected value of the lottery faced by all the types that deliver some given report. To illustrate, let us consider again the triplet of types $\{\theta^-,\theta,\theta^+\}$ and suppose that they all report θ. Under (3), the probability of receiving profit $\pi_1(\theta)$ is *increasing* across types θ^-, θ and θ^+. Conversely, the probability of receiving profit $\pi_n(\theta)$ is *decreasing* across types θ^-, θ and θ^+. Moreover, under (4), the profit variation faced by type θ^- in state 1, as it reports θ rather than truthtelling, is *greater* than that faced by type θ^+ with that same report. Conversely, the profit variation faced by type θ^- in state n is *smaller* than that faced by type θ^+.

[8] If $L = 0$, then limited liability constraints boil down to *ex-post* participation constraints. Looking at that case would prevent us from studying first-best implementation, which is, in that case, beyond reach.

3 Informal presentation of results

We begin by providing a heuristic presentation of our findings. To that end, we begin by stating the incentive constraint $(IC^{\theta}_{\theta'})$, whereby θ is not an attractive report to type θ' :

$$\sum_{s\in N} p_s(\theta')\pi_s(\theta') \geq \sum_{s\in N} p_s(\theta')\pi_s(\theta) + (\theta - \theta')\, q(\theta) + K(\theta) - K(\theta').$$

The impact of the signals is visible in the terms $\sum_{s\in N} p_s(\theta')\pi_s(\theta')$ and $\sum_{s\in N} p_s(\theta')\pi_s(\theta)$, which represent the expected value of the lottery faced by type θ', respectively, if it tells the truth and if it lies. Without signals, each of those terms would be replaced by a single profit, namely $\pi(\theta')$ and $\pi(\theta)$. The second term in the right-hand side of $(IC^{\theta}_{\theta'})$, namely $(\theta - \theta')\, q(\theta) + K(\theta) - K(\theta')$, represents the difference between true and fake cost incurred when the report θ is made. At first-best allocation, (2) holds for θ', and the constraint reduces to

$$\sum_{s\in N} p_s(\theta')\pi_s(\theta) \leq (\theta' - \theta)\, q^*(\theta) + K(\theta') - K(\theta)\,. \tag{5}$$

According to (5), the expected value of the lottery type θ' is faced with, if it reports θ, should not exceed the difference between true and fake cost. The value this difference takes depends not only on how θ' compares with θ but also on whether the cost is or not monotonic. Consequently, the lottery of profits to be designed for type θ will also depend on whether the cost is or not monotonic. This will becomes clearer as we present the two cases of monotonic and non-monotonic cost here below.

3.1 Monotonic cost

Both here and elsewhere in the study, when the cost is assumed to be monotonic, attention is restricted to the case in which the cost increases for all types. The reason is that this is tantamount to the "standard" case considered by the literature on contractual design with informative signals and limited liability. Formally, in our model: $q^*(\theta) + K'(\theta) > 0$, $\forall\theta$.

To analyse (5) with regards to types both below and above θ, we take any triplet of types $\{\theta^-, \theta, \theta^+\}$ including θ and such that $\theta^- < \theta < \theta^+$. When type θ^- reports θ, the difference between true and fake cost, namely $(\theta^- - \theta)\, q^*(\theta) + K(\theta^-) - K(\theta)$, takes a negative value. That is, types that exaggerate information *gain* in cost reimbursement. Hence, the expected profit faced by type θ^-, if it reports θ, must be negative. Under (2), the lottery of type θ should include at least one positive profit (a reward) and at least one negative profit (a punishment). Take such profits to be $\pi_1(\theta)$ and $\pi_n(\theta)$. With $\sum_{s\in N} p_s(\theta^-)\, \pi_s(\theta) < 0 = \sum_{s\in N} p_s(\theta)\, \pi_s(\theta)$ and $p'_1(\cdot) > 0 > p'_n(\theta)$, given (3), it must be the case that $\pi_1(\theta) > 0 > \pi_n(\theta)$. That is, type θ must be rewarded when the signal that is least likely to be drawn by type θ^- is realized; it must be punished when the signal that is the most likely to be drawn by type θ^- is realized. Obviously, type θ^+ is faced with the same profits, if it reports θ. With those profits, the lottery will come out to be favourable (rather than unfavourable) to type θ^+, since this type is more likely to draw signal 1 than signal n. However, this does not need be an issue. Indeed, when type θ^+ reports θ, the difference between true and fake cost, namely $\left(\theta^+ - \theta\right) q^*(\theta) + K(\theta^+) - K(\theta)$, takes a positive value. That is, types that understate information *lose* in cost reimbursement. Hence, the expected profit faced by type θ^-, if it reports θ, can be positive, in turn. This all yields the following pair of conditions:

$$\sum_{s \in N} p_s\left(\theta^-\right) \pi_s(\theta) < 0 < \sum_{s \in N} p_s\left(\theta^+\right) \pi_s(\theta).$$

A lottery with these characteristics can be made sufficiently unfavourable to type θ^-, while not being too favourable to type θ^+, as long as the gain in cost reimbursement accruing to type θ^-, if it reports θ, is not too high relative to the loss in cost reimbursement faced by type θ^+, following that same report. This requires the cost not being highly concave with respect to type.

3.2 Non-monotonic cost

To account for the possibility of the cost being non-monotonic with respect to type, we define $\widehat{\theta}$ such that $q^*(\widehat{\theta}) + K'(\widehat{\theta}) = 0$, as is usual in the literature on agency problems with countervailing incentives. In this section, we take $\widehat{\theta}$ to exist and to lie in the interior of the feasible set: $\widehat{\theta} \in \left(\underline{\theta}, \overline{\theta}\right)$.

Over the two ranges of types identified by $\widehat{\theta}$, the marginal cost has opposite signs that depend on the specific shape of the total cost. When the cost is U shaped with respect to type, as represented in graph (i) of Figure 1, $q^*(\theta) + K'(\theta) < 0$, $\forall \theta < \widehat{\theta}$, and $q^*(\theta) + K'(\theta) > 0$, $\forall \theta > \widehat{\theta}$. The converse is true when the cost is reverse U shaped, as represented in graph (ii) of Figure 1. It is thus not surprising that the sign of the cost difference in (5) will now depend both on how θ' compares with θ and on the shape of the total cost. To get a first clue on the consequences this all has in term of lottery design, it is useful to consider that two specific effects are at work thereof.

The first effect is that the structure of the lottery differs across types and, for each type, it depends on the shape of the cost. Consider, for instance, a U shaped total cost. On the decreasing side of the curve in graph (i) of Figure 1, type θ_3 gains in cost reimbursement, if it reports θ_1; type θ_2 loses, instead. Hence, type θ_3 should face a lottery with negative expected value, if it reports θ_1, which further involves that type θ_2 will rather face a lottery with positive expected value, if it delivers that same report. The converse occurs with types on the increasing side of the cost. For instance, type θ_5 should face a lottery with negative expected value, if it reports θ_4, which further involves that type θ_6 will face a lottery with positive expected value. It is thus evident that the lottery of a type on the decreasing side of the cost cannot have the same structure as the lottery of a type on the increasing side of the cost. In addition, for each given type, the structure of the lottery will also change if the cost is reverse U shaped, as represented in graph (ii) of Figure 1. Given that gains and losses in cost reimbursement are reversed between the two sides of the cost curve in that case, the incentives to cheat will be

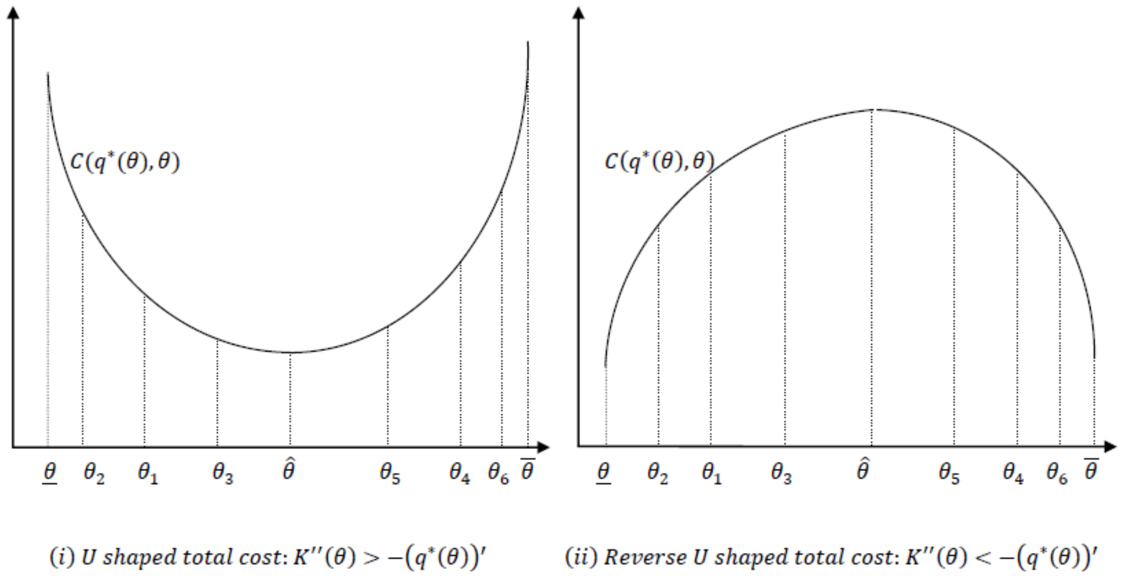


Figure 1: Total cost as a function of type

reversed as well.

The second and more intriguing effect is that, for any given type in the interior of the feasible set, either lower types or higher types may not display monotonic incentives to misrepresent information. Depending on the shape of the total cost, non-monotonicity may either facilitate or impose restrictions to the possibility of designing incentive compatible lotteries for some intermediate types. To clarify, it is useful to look again at the two cases in which the cost is U shaped and reverse U shaped with respect to type.

Let us first consider a U shaped cost. In graph (i) of Figure 1, some of the types above θ_1 gain in cost reimbursement if they claim θ_1. Such types are those in the range $(\theta_1, \widehat{\theta}]$, together with those in the range $(\widehat{\theta}, h(\theta_1))$, for some $h(\theta_1)$ that pins down the type which neither gains nor loses in terms of cost reimbursement, if it reports θ_1. When some types above $\widehat{\theta}$ do not gain in cost reimbursement, namely those in the range $\left[h(\theta_1), \overline{\theta}\right]$, if this range exists, the second effect is at work. Importantly, this effect facilitates the lottery design. First, due to the countervailing effect induced by the opportunity cost, the types above $\widehat{\theta}$ that gain in cost reimbursement if they report θ_1, gain less than type $\widehat{\theta}$. Hence, the lottery any such type faces when reporting θ_1 will be sufficiently unattractive, if it is so to type $\widehat{\theta}$. Second, the types above $\widehat{\theta}$ that lose in cost reimbursement, if they report θ_1, have no interest in delivering that report, since they would then face a lottery with negative expected value, as any other type above θ_1. Analogous reasoning applies, *mutatis mutandis*, to type θ_4, and the types below $\widehat{\theta}$ that may claim θ_4.

A different conclusion is reached when the cost is reverse U shaped in that the second effect complicates the lottery design rather than facilitating it. Consider, for instance, type θ_1 in graph (ii) of Figure 1. As the lottery of θ_1 must discourage lower types from exaggerating information, it will have a positive expected value to higher types claiming θ_1. This is not an issue as long as types immediately above θ_1 are concerned, provided those types lose in cost reimbursement, if they report θ_1. However, due to the countervailing effect triggered by the opportunity cost, there are types above $\widehat{\theta}$ which gain in cost reimbursement, if they report θ_1. These are the types in the range $\left[h\left(\theta_1\right), \overline{\theta}\right]$, if it exists. For instance, one such type might be θ_5 in graph (ii) of Figure 1. The lottery of θ_1 being favorable to those types, they will enjoy a double benefit by claiming θ_1. In that case, it is impossible for the principal to make the lottery of type θ_1 incentive compatible for all the other types, while also retaining all surplus from type θ_1. For analogous reason, some types below $\widehat{\theta}$ may enjoy a double benefit when they report a type like θ_4 and, again, it may be impossible to design an effective lottery for type θ_4. This kind of difficulty arises with types on either side of the total cost when this cost is sharp sloping (the opportunity cost is concave). Then, unlike with a U shaped total cost, the second effect may prevent first-best implementation. When the total cost is smooth sloping (the opportunity cost is slightly convex), instead, the second effect is *not* at work and, as with a U shaped total cost, incentive compatible lotteries can be designed to attain the first-best outcome.

We are now ready to develop the formal analysis and highlight the exact mechanics through which the two effects of non-monotonicity determine the contractual attainments.

4 The lottery design

We proceed as follows. First, we reformulate the agent's incentive constraints to make it apparent how the properties of the probabilities of the signals affect the way in which the principal can attain incentive compatibility under limited liability. Next, we restate the principal's programme accordingly and we investigate how the lottery should be structured to solve it. Lastly, we highlight the specific effects of non-monotonicity on the lottery design.

Being based on (2), we obtain an expression of $\pi_1(\theta)$, which can then be used to derive a new formulation of (5). Recall that (5) is the incentive constraint whereby θ is not an attractive report to type θ'. Using (3) and taking again the triplet $\left\{\theta^-, \theta, \theta^+\right\}$, the new formulation specifies in the following two conditions, respectively, for $\theta' = \theta^-$ and $\theta' = \theta^+$ (details are provided in Appendix A.1):

$$\pi_n(\theta) \leq \frac{\theta - \theta^-}{-p_n(\theta)} \frac{q^*(\theta) + \frac{K(\theta) - K\left(\theta^-\right)}{\theta - \theta^-}}{\frac{p_n(\theta^-)}{p_n(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}} - \sum_{s \neq 1,n} \pi_s(\theta) \frac{p_s(\theta)}{p_n(\theta)} \frac{\frac{p_s(\theta^-)}{p_s(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}}{\frac{p_n(\theta^-)}{p_n(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}} \tag{6}$$

$$\pi_n(\theta) \geq \frac{\theta^+ - \theta}{-p_n(\theta)} \frac{q^*(\theta) + \frac{K\left(\theta^+\right) - K(\theta)}{\theta^+ - \theta}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} - \sum_{s \neq 1,n} \pi_s(\theta) \frac{p_s(\theta)}{p_n(\theta)} \frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}}. \tag{7}$$

Under (6), θ is not an attractive report to lower types; under (7), θ is not an attractive report to higher types. Joint inspection of (6) and (7) highlights why it is useful to work with this pair of conditions, instead of dealing with the single incentive constraint. Indeed, this "duplication" shows how the profit should be chosen in state n for any "intermediate" type θ to attract lies neither from lower types nor from higher types. The higher that $\pi_n(\theta)$ is the more that type θ^- is eager to claim θ. On the other hand, the lower that $\pi_n(\theta)$ is the more that type θ^+ is eager to claim θ. Therefore, types θ^- and θ^+ are both unwilling to announce θ only if the value of $\pi_n(\theta)$ is set neither too low nor too high. Taking the limit of the right-hand side of (6) and (7), respectively, as $\theta^- \to \theta$ and $\theta^+ \to \theta$, we see that the two conditions hold jointly if and only if:

$$\pi_n(\theta) = \frac{q^*(\theta) + K'(\theta) + \sum_{s\neq 1,n} \pi_s(\theta) p_s(\theta) \left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_s'(\theta)}{p_s(\theta)}\right)}{-p_n(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right)}. \tag{8}$$

This is the state$-n$ profit such that any incentives to mimic a *neighboring* type are eliminated. Using (8), (6) and (7) are respectively reformulated as follows:

$$\begin{aligned} q^*(\theta) + K'(\theta) \geq \left(\theta - \theta^-\right) \left(q^*(\theta) + \frac{K(\theta) - K\left(\theta^-\right)}{\theta - \theta^-}\right) \frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}}{\frac{p_n(\theta^-)}{p_n(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}} \\ + \sum_{s\neq 1,n} \pi_s(\theta) p_s(\theta) \left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right) \left(\frac{\frac{p_s(\theta^-)}{p_s(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}}{\frac{p_n(\theta^-)}{p_n(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}} - \frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_s'(\theta)}{p_s(\theta)}}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}} \right) \end{aligned} \tag{9}$$

and

$$\begin{aligned} q^*(\theta) + K'(\theta) \leq \left(\theta^+ - \theta\right) \left(q^*(\theta) + \frac{K\left(\theta^+\right) - K(\theta)}{\theta^+ - \theta}\right) \frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} \\ + \sum_{s\neq 1,n} \pi_s(\theta) p_s(\theta) \left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right) \left(\frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} - \frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_s'(\theta)}{p_s(\theta)}}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}} \right). \end{aligned} \tag{10}$$

We see that (9) and (10) hold jointly only if

$$\begin{aligned} \left(\theta^+ - \theta\right) \frac{q^*(\theta) + \frac{K\left(\theta^+\right) - K(\theta)}{\theta^+ - \theta}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} - \left(\theta - \theta^-\right) \frac{q^*(\theta) + \frac{K(\theta) - K\left(\theta^-\right)}{\theta - \theta^-}}{\frac{p_n(\theta^-)}{p_n(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}} \\ \leq \sum_{s\neq 1,n} \pi_s(\theta) p_s(\theta) \left(\frac{\frac{p_s(\theta^-)}{p_s(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}}{\frac{p_n(\theta^-)}{p_n(\theta)} - \frac{p_1(\theta^-)}{p_1(\theta)}} - \frac{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_s(\theta^+)}{p_s(\theta)}}{\frac{p_1(\theta^+)}{p_1(\theta)} - \frac{p_n(\theta^+)}{p_n(\theta)}} \right). \end{aligned} \tag{11}$$

In the limit, (11) reduces to (9) as $\theta^- \to \theta$ and to (10) as $\theta^+ \to \theta$. The incentive constraints can thus be replaced by the local incentive constraint (8), which must hold for all θ, together with (11), which must hold for all triplets $\left\{\theta^-, \theta, \theta^+\right\}$. In what follows, we refer to (11) as to the global incentive constraint.

Taking this all into account, and provided that the agent retains no surplus to deliver the first-best quantity, the principal's programme is reformulated as follows:

$$\underset{\boldsymbol{\pi}(\theta),\forall\theta}{Max} \int_{\underline{\theta}}^{\overline{\theta}} \left(S\left(q^*\left(\theta\right)\right) - C\left(q^*\left(\theta\right), \theta\right)\right) dF\left(\theta\right)$$

$$\text{subject to}$$

$$(2)\,, \quad (8)\,, \quad (11) \text{ and } \left(LL_\theta^s\right).$$

We now need to understand how the lottery is to be structured for any type θ so that all the constraints in the programme are satisfied.

For the time being, we neglect global incentive compatibility and look for the locally incentive compatible lottery under which limited liability constraints are weakest. First take type θ to be such that $q^*(\theta) + K'(\theta) > 0$. Recall that this is the case of all types when the cost is monotonically increasing everywhere. To satisfy the local incentive constraint (8), it should be considered that the types below θ gain in cost reimbursement, if they report θ. With that report, such types are less likely to receive $\pi_1(\theta)$ than any other profit. Therefore, to discourage them from announcing θ, the principal should set $\pi_1(\theta)$ to be the highest profit of type θ, hence a reward. Under (2), with $\pi_1(\theta)$ being a reward, at least one of the remaining profits should be a punishment. In fact, limited liability constraints are weakest if the remaining profits are all punishments, set such that $\pi_s(\theta) = \pi_n(\theta)$, $\forall s \neq 1, n$. Suppose that, for some given $s \neq 1, n$, $\pi_s(\theta)$ and $\pi_n(\theta)$ are not both punishments or, more generally, their values are different. Then, there is room for relaxing limited liability constraints by adjusting profits in such a way that (8) still holds. For instance, if $\pi_n(\theta) < \pi_s(\theta)$, then (LL_θ^n) could be relaxed by raising $\pi_n(\theta)$. In turn, $\pi_s(\theta)$ should be decreased to preserve local incentive compatibility, and $\pi_1(\theta)$ should be adjusted to also secure surplus extraction. The locally incentive compatible lottery which relaxes limited liability constraints to the utmost is thus derived. This lottery, denoted $\boldsymbol{\pi}^\alpha(\theta)$, is structured as follows (see Appendix A.2 for mathematical details):

$$\pi_1(\theta) = \left(q^*(\theta) + K'(\theta)\right) \frac{1 - p_1(\theta)}{p_1'(\theta)} \tag{12}$$

$$\pi_s(\theta) = \left(q^*(\theta) + K'(\theta)\right) \frac{p_1(\theta)}{-p_1'(\theta)}, \ \forall s \neq 1. \tag{13}$$

Next take type θ to be such that $q^*(\theta) + K'(\theta) < 0$. To satisfy (8), which is now conveniently reformulated in terms of $\pi_1(\theta)$, rather than of $\pi_n(\theta)$, it should be considered that the types which might want to report θ are those above θ in this case. Under (3), those types are less likely to draw signal n than any other signal. Hence, n is the state in which type θ should be assigned the highest profit (a reward). In all other states, it should face an equal punishment, instead. This is the incentive compatible lottery which weakens limited liability constraints to the utmost. Formally, the lottery, denoted $\boldsymbol{\pi}^{\beta}(\theta)$, is composed as follows (see Appendix A.3):

$$\begin{aligned}
\pi_n(\theta) &= (q^*(\theta) + K'(\theta)) \frac{1 - p_n(\theta)}{p_n'(\theta)} && (14) \\
\pi_s(\theta) &= (q^*(\theta) + K'(\theta)) \frac{p_n(\theta)}{-p_n'(\theta)}, \ \forall s \neq n. && (15)
\end{aligned}$$

Lemma 1 (LL_{θ}^{s}) *is satisfied for all* $s \in N$ *if and only if it is satisfied for* $s = n$ *by lottery* $\boldsymbol{\pi}^{\alpha}(\theta)$, *when* θ *is such that* $q^*(\theta) + K'(\theta) > 0$, *and for* $s = 1$ *by lottery* $\boldsymbol{\pi}^{\beta}(\theta)$, *when* θ *is such that* $q^*(\theta) + K'(\theta) < 0$.

This result clarifies that it is not possible for the principal to design a locally incentive compatible lottery that satisfies limited liability constraints in any possible contingency, if $\boldsymbol{\pi}^{\alpha}(\theta)$ and $\boldsymbol{\pi}^{\beta}(\theta)$ fail to do so, respectively, for θ such that $q^*(\theta) + K'(\theta) > 0$ and θ such that $q^*(\theta) + K'(\theta) < 0$. Nonetheless, one should not conclude that the limited liability constraint is necessarily binding at least in some state when those lotteries are used. Actually, depending on the magnitude of L, it may be the case that (LL_{θ}^{s}) is slack for any s. Under this circumstance, insisting on those lotteries restricts the possibility, for the principal, of taking advantage of correlated information to enhance contracting. This suggests that there might be scope for adopting a different lottery of profits, if $\boldsymbol{\pi}^{\alpha}(\theta)$ and $\boldsymbol{\pi}^{\beta}(\theta)$ fail to satisfy the global incentive constraint (11).

Unlike the local incentive constraint (8), the global incentive constraint (11) is relaxed to the utmost when both the profit assigned in state 1 and the profit assigned in state n, rather than only one of them, are set to differ from the profit assigned in all the other states. Indeed, the principal can take greater advantage of the correlation between signal and type, if she saturates (LL_{θ}^{s}), $\forall s \neq 1, n$, and then uses the profit in state n to satisfy (8) and adjusts the profit in state 1 to retain all surplus. For any type θ this lottery, to be denoted $\boldsymbol{\pi}^{\gamma}(\theta)$, looks as follows (see Appendix A.4):

$$\begin{aligned}
\pi_1(\theta) &= \frac{q^*(\theta) + K'(\theta) - L\frac{p_n'(\theta)}{p_n(\theta)}}{p_1(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right)} - L && (16) \\
\pi_n(\theta) &= \frac{L\frac{p_1'(\theta)}{p_1(\theta)} - (q^*(\theta) + K'(\theta))}{p_n(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right)} - L && (17) \\
\pi_s(\theta) &= -L, \ \forall s \neq 1, n. && (18)
\end{aligned}$$

Lemma 2 *For any* θ, *there exists a lottery of profits for such that* (11) *is satisfied jointly with* (LL_{θ}^{s}), $\forall s \in N$, *if and only if this is the case of lottery* $\boldsymbol{\pi}^{\gamma}(\theta)$.

Noticeably, unlike the profits previously characterized (recall Lemma 1), those in lottery $\boldsymbol{\pi}^{\gamma}(\theta)$ apply regardless of the sign of $q^*(\theta) + K'(\theta)$, for the following reason. As far as types in a neighborhood of θ are concerned, the incentives to report θ are relevant in only one direction. That is, either only types immediately below θ might be attracted by the report θ, or only types immediately above θ might be attracted by that report. By contrast, when it comes to consider more distant types, the incentives to report θ are relevant both downwards and upwards. Hence, the global incentive constraint must be verified for types both below and above θ. To see what this all entails, first suppose that θ is such that $q^*(\theta)+K'(\theta)>0$. Then, the types which might want to report θ are those immediately below θ, which would gain in cost reimbursement. In this case, it is beneficial to reward type θ in state 1 because signal 1 is least likely to be drawn by those types. Next suppose that θ is such that $q^*(\theta)+K'(\theta)<0$. Then, the types which might want to report θ are those immediately above θ. It is now beneficial to reward type θ in state n because signal n is least likely to be drawn by those types. In substance, profits in states 1 and n are conveniently chosen to satisfy the local incentive constraint jointly with (2) because one signal between 1 and n is least likely to be drawn by any type which might have an incentive to report θ. Once this is done, it remains to set the other profits in such a way that the global incentive constraint (11) is satisfied as well.

In definitive, depending on how stringent the local incentive constraint is, the contractual offer will include one of the lotteries $\boldsymbol{\pi}^{\alpha}(\theta)$ and $\boldsymbol{\pi}^{\gamma}(\theta)$ for θ such that $q^*(\theta)+K'(\theta)>0$, and one of the lotteries $\boldsymbol{\pi}^{\beta}(\theta)$ and $\boldsymbol{\pi}^{\gamma}(\theta)$ for θ such that $q^*(\theta)+K'(\theta)<0$.

4.1 The effects of non-monotonicity on the lottery choice

To explain how non-monotonicity of the cost affects the lottery choice, we first return to the monotonic case, focusing again on a positive marginal cost for all types. Recall that in our model this is the case when $q^*(\theta)+K'(\theta)>0$, $\forall\theta$. From previous lemmas we deduce that, in this case, the principal should choose between $\boldsymbol{\pi}^{\alpha}(\theta)$ and $\boldsymbol{\pi}^{\gamma}(\theta)$. As $\boldsymbol{\pi}^{\alpha}(\theta)$ is the locally incentive compatible lottery which relaxes limited liability constraints to the utmost (Lemma 1), it should be chosen when limited liability is the main concern. In turn, as $\boldsymbol{\pi}^{\gamma}(\theta)$ is the lottery which relaxes global incentive constraints to the utmost under limited liability (Lemma 2), it is preferable when the main concern is global incentive compatibility, and limited liability is not too tight. Therefore, the choice between $\boldsymbol{\pi}^{\alpha}(\theta)$ and $\boldsymbol{\pi}^{\gamma}(\theta)$ depends on how concerning limited liability and global incentive compatibility are.

To examine the lottery choice in depth, it is useful to have a look at the way in which the use of lotteries helps the principal extract surplus. By announcing θ, type θ^- gains an amount of $\left(\theta-\theta^-\right)q^*(\theta)+K(\theta)-K\left(\theta^-\right)$ in terms of cost reimbursement but it also incurs a penalty equal to the expected value of the lottery, which is negative:

$$\begin{aligned}
\sum_s \pi_s(\theta)\,p_s\left(\theta^-\right) &= \pi_n(\theta)\sum_{s\neq 1} p_s(\theta)\left(\frac{p_s(\theta^-)}{p_s(\theta)}-\frac{p_1(\theta^-)}{p_1(\theta)}\right)\\
&= \pi_n(\theta)\,\frac{p_1(\theta)-p_1(\theta^-)}{p_1(\theta)}.
\end{aligned}$$

On the other hand, by announcing θ, type θ^+ loses an amount of $\left(\theta^+-\theta\right)q^*(\theta)+K\left(\theta^+\right)-K(\theta)$ in terms of cost reimbursement but it also obtains a gain equal to the expected value of the lottery, which is now positive:

$$\sum_s \pi_s(\theta) p_s(\theta^+) = \pi_n(\theta) \sum_{s \neq 1} p_s(\theta) \left(\frac{p_s(\theta^+)}{p_s(\theta)} - \frac{p_1(\theta^+)}{p_1(\theta)} \right)$$
$$= \pi_n(\theta) \frac{p_1(\theta) - p_1(\theta^+)}{p_1(\theta)}.$$

With $p_1(\cdot)$ being concave under (4), the principal can use $\boldsymbol{\pi}^\alpha(\theta)$ if the ratio between the penalty incurred by type θ^- and the gain obtained by type θ^+ in terms of lottery is greater than the ratio between the gain obtained by type θ^- and the penalty incurred by type θ^+ in terms of cost reimbursement. This is the case when the following condition is satisfied:

$$\frac{q^*(\theta) + \frac{K(\theta) - K(\theta^-)}{\theta - \theta^-}}{q^*(\theta) + \frac{K(\theta^+) - K(\theta)}{\theta^+ - \theta}} \leq \frac{\frac{p_1(\theta) - p_1(\theta^-)}{\theta - \theta^-}}{\frac{p_1(\theta^+) - p_1(\theta)}{\theta^+ - \theta}}. \tag{19}$$

Under (19), there exists a negative value of $\pi_n(\theta)$ such that neither type θ^- nor type θ^+ finds it convenient to announce θ. When (19) does not hold but the limited liability constraints are not saturated, the principal should renounce to $\boldsymbol{\pi}^\alpha(\theta)$ in favour of $\boldsymbol{\pi}^\gamma(\theta)$. This would enable the principal to inflict higher punishments and, hence, to relax the conflict between upward and downward incentive constraints to the utmost.

Let us now turn to the case in which the cost is non-monotonic with respect to type. Recall that the marginal cost $q^*(\theta) + K'(\theta)$ is now positive for all θ on one side of $\widehat{\theta}$ and negative for all θ on the other side. We identify *two specific effects* of non-monotonicity on contractual design.

First, as regards types such that $q^*(\theta) + K'(\theta) < 0$ (decreasing cost), the principal has to choose between $\boldsymbol{\pi}^\beta(\theta)$ and $\boldsymbol{\pi}^\gamma(\theta)$ (rather than between $\boldsymbol{\pi}^\alpha(\theta)$ and $\boldsymbol{\pi}^\gamma(\theta)$) because those types gain in cost reimbursement, if they understate (rather than overstating) information. Thus, the condition under which upward and downward incentive constraints are jointly satisfied for any triplet $\{\theta^-, \theta, \theta^+\}$ is specified as follows:

$$\frac{\frac{p_n(\theta) - p_n(\theta^-)}{\theta - \theta^-}}{\frac{p_n(\theta^+) - p_n(\theta)}{\theta^+ - \theta}} \leq \frac{q^*(\theta) + \frac{K(\theta) - K(\theta^-)}{\theta - \theta^-}}{q^*(\theta) + \frac{K(\theta^+) - K(\theta)}{\theta^+ - \theta}}. \tag{20}$$

When (20) holds, there exists a negative value of $\pi_1(\theta)$ such that θ is an attractive report neither to type θ^- nor to type θ^+. Once again, when (20) fails to hold and provided that limited liability constraints are not saturated, the principal should rather rely on $\boldsymbol{\pi}^\gamma(\theta)$. Overall, the choice is between $\boldsymbol{\pi}^\alpha(\theta)$ and $\boldsymbol{\pi}^\gamma(\theta)$ for θ such that $q^*(\theta) + K'(\theta) > 0$ and between $\boldsymbol{\pi}^\beta(\theta)$ and $\boldsymbol{\pi}^\gamma(\theta)$ for types such that $q^*(\theta) + K'(\theta) < 0$. As shown by MRC, it is the shape of $K(\cdot)$ that determines the ranges of types with increasing and decreasing cost. In our case, the shape of $K(\cdot)$ determines whether $\boldsymbol{\pi}^\alpha(\cdot)$ or $\boldsymbol{\pi}^\beta(\cdot)$ should be used for the types in the two ranges, provided that (19) and (20) hold.

Second, whereas with a monotonic cost all types below θ gain in cost reimbursement, if they pretend θ, and all types above θ lose instead, this may not be the case with a non-monotonic cost. To see this, it is useful to define $h(\theta)$ such that:

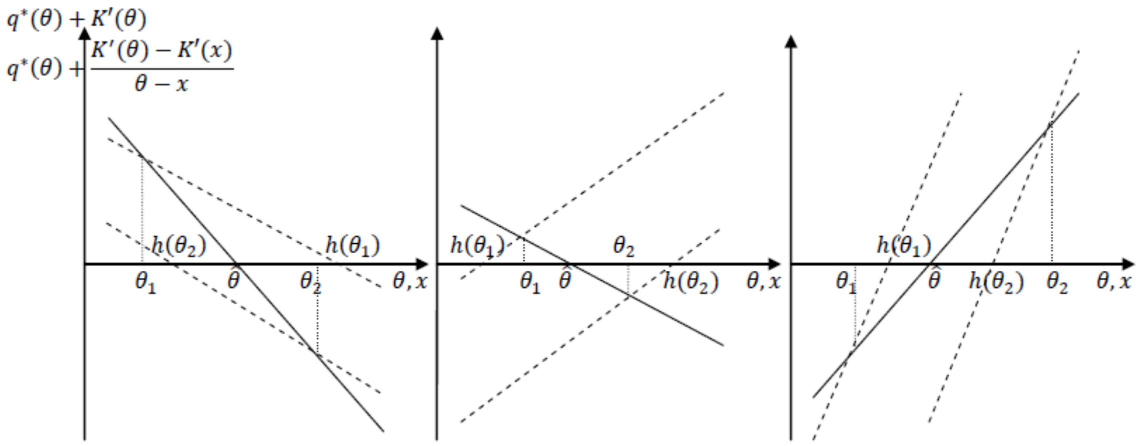
$$q^*(\theta) + \frac{K(\theta) - K(h(\theta))}{\theta - h(\theta)} = 0. \tag{21}$$

That is, $h(\theta)$ is the type which neither gains nor loses in terms of cost reimbursement, if it reports θ. Importantly, $h(\theta)$ may or may not coincide with some type in the *interior* of the feasible set. When $h(\theta) \in (\underline{\theta}, \overline{\theta})$ the incentives to misrepresent information are not monotonic across all types below θ and across all types above θ. First take θ such that $q^*(\theta)+K'(\theta) > 0$. By announcing θ, type θ^+ loses in cost reimbursement if $\theta^+ > h(\theta)$, whereas it gains if $\theta^+ < h(\theta)$. Take now $q^*(\theta)+K'(\theta) < 0$. By announcing θ, type θ^+ incurs a penalty in cost reimbursement if $\theta^+ < h(\theta)$ and a gain if $\theta^+ > h(\theta)$. Similar considerations can be made for types θ^-. This effect imposes restrictions on the design of incentive compatible lotteries. For instance, when $q^*(\theta)+K'(\theta) < 0$ a lottery designed to extract the gain in cost reimbursement from types below θ may attract lies from types above $h(\theta)$, which gain in both cost reimbursement and lottery by announcing θ. To ascertain whether all types below (above) θ get a bonus by reporting θ, or they all lose, it is necessary to assess how $h(\theta)$ compares with θ. The next result shows that this is related to the shape of the cost (the proof is in Appendix B).

Lemma 3 *If either $K''(\theta) < 0$ or $K''(\theta) \geq -(q^*(\theta))'$, $\forall\theta$, then $h(\theta) > \theta$ if and only if $\theta < \widehat{\theta}$. If $0 < K''(\theta) < -(q^*(\theta))'$, $\forall\theta$, then $h(\theta) < \theta$ if and only if $\theta < \widehat{\theta}$.*

The various cases identified by the lemma are better understood by looking at the graphs in Figure 4.1. In each graph, the thick line represents $q^*(\theta) + K'(\theta)$ as a function of θ. Each of the two dashed lines represents $q^*(\theta) + \frac{K(\theta)-K(x)}{\theta-x}$ as a function of $x \in [\underline{\theta}, \overline{\theta}]$ for some given value of θ, taken to be $\theta_1 < \widehat{\theta}$ for the upper line and $\theta_2 > \widehat{\theta}$ for the other. These are the values of θ at which the thick line and each of the two dashed lines cross. The values of x at which the dashed lines cross the horizontal axis are $h(\theta_1)$ and $h(\theta_2)$. Graph (i) shows that when the opportunity cost is concave, by reporting $\theta_1 < \widehat{\theta}$, any type $x < \theta_1$ and any type $x \in (\theta_1, h(\theta_1))$ gains in cost reimbursement because $q^*(\theta_1) + \frac{K(\theta_1)-K(x)}{\theta_1-x} > 0$ for all types in those ranges; by contrast, any type $x > h(\theta_1)$ loses in cost reimbursement because $q^*(\theta_1) + \frac{K(\theta_1)-K(x)}{\theta_1-x} < 0$. By reporting $\theta_2 > \widehat{\theta}$, any type $x < h(\theta_2)$ gains because $q^*(\theta_2) + \frac{K(\theta_2)-K(x)}{\theta_2-x} > 0$, whereas any type $x > h(\theta_2)$, whether below or above θ_2, loses because $q^*(\theta_2) + \frac{K(\theta_2)-K(x)}{\theta_2-x} < 0$. Graphs (ii) and (iii) are interpreted in a similar manner, *mutatis mutandis*.

Taking the two effects together, it is not surprising that the shape of $K(\cdot)$ determines the principal's contractual attainments, which we now turn to present.



(i) $K(\cdot)$ *concave* (ii) $K(\cdot)$ *slightly convex* (iii) $K(\cdot)$ *highly convex*

5 Conditions for first-best implementation

5.1 $K(\cdot)$ convex

When the opportunity cost is convex, regardless of the exact degree of convexity, it is not an issue for the principal to attain global incentive compatibility. This finding is formalized here below (and proved in Appendix C).

Lemma 4 *If* $K''(\cdot) \geq 0$, *then* (11) *is satisfied* $\forall\theta$.

It follows directly from the lemma that the condition for global incentive compatibility is weakest with either $\boldsymbol{\pi}^{\alpha}(\theta)$ or $\boldsymbol{\pi}^{\beta}(\theta)$, depending on the sign of the marginal cost and, hence, on the shape of the total cost.

First take $K(\cdot)$ to be slightly convex $\left(0 \leq K''(\cdot) < -(q^*(\cdot))'\right)$, entailing that the total cost is reverse U shaped with respect to type. We saw that non-monotonicity of the cost induces two potential effects on contractual design. However, the second effect is not at work in this case. Take any intermediate type θ. As the marginal opportunity cost varies little with type, all types below θ gain in cost reimbursement, if they report θ, whereas all types above θ lose. Consider, for instance, $\theta < \widehat{\theta}$ and some type θ^+ above θ. If $\theta^+ < \widehat{\theta}$, then type θ^+ loses in cost reimbursement, if it reports θ, provided the total cost is increasing for both θ and θ^+. If $\theta^+ > \widehat{\theta}$, instead, then type θ^+ is penalized in cost reimbursement over the range $(\widehat{\theta}, \theta^+)$, where the marginal gain in opportunity cost is lower than the marginal loss in production cost; it obtains a prize in cost reimbursement over the range $(\theta, \widehat{\theta})$, where the converse occurs. With $K(\cdot)$ slightly convex, the marginal effect associated with the opportunity cost is small, hence so is the prize in cost reimbursement as well. Therefore, type θ^+ incurs a net penalty, if it claims θ, as is the case if the total cost increases for all types. Formally, $h(\theta)$ does not lie within the set of feasible types. Being based on a similar argument, one can explain why, for any $\theta > \widehat{\theta}$, types below θ are all penalized in cost reimbursement, if they report θ, even if the total cost is increasing. Ruled out the second effect, it remains to verify that (19) is satisfied for all $\theta < \widehat{\theta}$ and (20) for all $\theta > \widehat{\theta}$. One can easily check that this is true, indeed, given the convexity of $K(\cdot)$ and the concavity of $p_1(\cdot)$ and $p_n(\cdot)$ according to (4). As already explained, under these circumstances, $\boldsymbol{\pi}^{\alpha}(\theta)$ and $\boldsymbol{\pi}^{\beta}(\theta)$ are the most effective lotteries at extracting any gains in cost reimbursement, respectively, from types with increasing and decreasing cost.

Next take $K(\cdot)$ to be highly convex ($K''(\cdot) > -(q^*(\cdot))'$), entailing that the total cost is U shaped with respect to type. Then, the task of screening types is even easier for the principal. Consider again $\theta < \widehat{\theta}$ and some type $\theta^+ > \theta$. If $\theta^+ < \widehat{\theta}$, then, obviously, type θ^+ obtains a prize in cost reimbursement, if it reports θ, because the marginal gain in production cost exceeds the marginal loss in opportunity cost. If $\theta^+ > \widehat{\theta}$, instead, two contrasting effects are again to be considered, as with $K(\cdot)$ slightly convex. Particularly, in this case, type θ^+ obtains a prize in cost reimbursement over the range $(\theta, \widehat{\theta})$, where the marginal gain in production cost exceeds the marginal loss in opportunity cost; it is penalized over the range $(\widehat{\theta}, \theta^+)$, where the converse occurs. For high types, such that $\theta^+ > h(\theta)$, the loss induced by the opportunity cost is sufficiently high that reporting θ yields a net penalty in cost reimbursement. Therefore, lottery $\boldsymbol{\pi}^{\beta}(\theta)$, as designed to discourage types in the range $(\theta, h(\theta)]$ from claiming θ, is *a fortiori* effective with types in the range $\left(h(\theta), \overline{\theta}\right]$, provided those types lose in cost reimbursement, if they announce θ. Analogously, when $\theta > \widehat{\theta}$, the lottery $\boldsymbol{\pi}^{\alpha}(\theta)$ is especially effective at discouraging types in the range $[\underline{\theta}, h(\theta))$ from pretending θ, provided those types lose in cost reimbursement, if they announce θ. The following result is obtained (the proof is in Appendix D).

Proposition 1 *Assume that $K''(\cdot) \geq 0$. First best is implemented if and only if:*

$$(q^*(\theta) + K'(\theta)) \frac{p_1(\theta)}{p_1'(\theta)} \leq L, \ \forall\theta \text{ such that } q^*(\theta) + K'(\theta) > 0 \tag{22}$$

$$(q^*(\theta) + K'(\theta)) \frac{p_n(\theta)}{p_n'(\theta)} \leq L, \ \forall\theta \text{ such that } q^*(\theta) + K'(\theta) < 0. \tag{23}$$

This result compares with Proposition 2 in Gary-Bobo and Spiegel [10]. They assume that the agent's cost is increasing and convex for all types. In that case, in spite of the agent being protected by limited liability, first best is at hand, provided that local incentive constraints hold. When the cost is non-monotonic, but still convex with respect to type, there is only one novel aspect to that finding: local incentive compatibility requires targeting a different lottery to the types with decreasing cost.

5.2 $K(\cdot)$ concave

We now take $K''(\cdot) < 0$. Although the total cost is reverse U shaped as in the case of $K(\cdot)$ slightly convex, the effect of variations in the opportunity cost is now important and the second effect may be at work. When it is so, unlike in the case of $K(\cdot)$ highly convex, it worsens contracting, imposing restrictions on first-best implementation. Recalling the example with $\theta < \widehat{\theta}$ and $\theta^+ > \widehat{\theta}$, the prize in cost reimbursement accruing to type θ^+ over the range $(\theta, \widehat{\theta})$, if it announces θ, is not necessarily lower than the penalty incurred over the range $(\widehat{\theta}, \theta^+)$.

Actually, this is *not* the case whenever θ^{+} belongs to the range $\left(h(\theta), \overline{\theta}\right]$, if this range exists. Then, $\boldsymbol{\pi}^{\alpha}(\theta)$ fails to be incentive compatible because, by claiming θ, the types in that range would gain in cost reimbursement and, in addition, they would also face a favorable lottery. Analogously, the lottery $\boldsymbol{\pi}^{\beta}(\theta)$ assigned to $\theta > \widehat{\theta}$ is not incentive compatible for the types in the range $[\underline{\theta}, h(\theta))$, if this range exists, because such types would all obtain a double benefit by reporting θ. In either case, the principal extract all surplus without triggering lies, only if $\boldsymbol{\pi}^{\gamma}(\theta)$ is adopted. The following lemma summarizes these results (see Appendix E for the proof).

Lemma 5 *Suppose that* $K''(\cdot) < 0$.

(i) $\theta < \widehat{\theta}$ *and* $h(\theta) \notin \left(\underline{\theta}, \overline{\theta}\right)$: *Full surplus extraction is attained if and only if this is the case with lottery* $\boldsymbol{\pi}^{\alpha}(\theta)$ *when* (19) *holds, and with lottery* $\boldsymbol{\pi}^{\gamma}(\theta)$ *otherwise.*

(ii) $\theta > \widehat{\theta}$ *and* $h(\theta) \notin \left(\underline{\theta}, \overline{\theta}\right)$: *Full surplus extraction is attained if and only if this is the case with lottery* $\boldsymbol{\pi}^{\beta}(\theta)$ *when* (20) *holds, and with lottery* $\boldsymbol{\pi}^{\gamma}(\theta)$ *otherwise.*

(iii) $h(\theta) \in \left(\underline{\theta}, \overline{\theta}\right)$: *Full surplus extraction is attained if and only if this is the case with lottery* $\boldsymbol{\pi}^{\gamma}(\theta)$.

In substance, $\boldsymbol{\pi}^{\gamma}(\theta)$ is more effective than a lottery yielding the same punishment in all states but one not only when $K(\cdot)$ is too concave to have (19) or (20) satisfied, but also when non-monotonicity of the total cost makes global incentive compatibility difficult to attain (formally, when $h(\theta) \in \left(\underline{\theta}, \overline{\theta}\right)$). We now derive conditions under which first best is implemented, according to the optimal lotteries to be targeted to different ranges of types for different degrees of concavity of $K(\cdot)$ (mathematical details are found in Appendix F).

Proposition 2 *Assume that* $K''(\cdot) < 0$. *First best is implemented if and only if either:*

(i) (19) *and* (22) *hold for* $\theta \leq \widehat{\theta}$, (20) *and* (23) *hold for* $\theta \geq \widehat{\theta}$, *and* $h(\theta) \notin \left(\underline{\theta}, \overline{\theta}\right)$;

or

(ii) the following condition holds for all triplets $\left\{\theta^{-}, \theta, \theta^{+}\right\}$:

$$\frac{q^{*}(\theta) + \frac{K(\theta)-K(\theta^{-})}{\theta-\theta^{-}}}{\frac{p_1(\theta)-p_1(\theta^{-})}{\left(\theta-\theta^{-}\right)p_1(\theta)} - \frac{p_n(\theta)-p_n(\theta^{-})}{\left(\theta-\theta^{-}\right)p_n(\theta)}} - \frac{q^{*}(\theta) + \frac{K(\theta^{+})-K(\theta)}{\theta^{+}-\theta}}{\frac{p_1(\theta^{+})-p_1(\theta)}{\left(\theta^{+}-\theta\right)p_1(\theta)} - \frac{p_n(\theta^{+})-p_n(\theta)}{\left(\theta^{+}-\theta\right)p_n(\theta)}} \tag{24}$$

$$\leq L \sum_{s \neq 1,n} p_s(\theta) \left(\frac{\frac{p_1(\theta^{-})}{p_1(\theta)} - \frac{p_s(\theta^{-})}{p_s(\theta)}}{\frac{p_1(\theta^{-})}{p_1(\theta)} - \frac{p_n(\theta^{-})}{p_n(\theta)}} - \frac{\frac{p_1(\theta^{+})}{p_1(\theta)} - \frac{p_s(\theta^{+})}{p_s(\theta)}}{\frac{p_1(\theta^{+})}{p_1(\theta)} - \frac{p_n(\theta^{+})}{p_n(\theta)}} \right).$$

Proposition 2 enriches Proposition 1 in Danau and Vinella [7] by considering also the case in which the agent's total cost is non-monotonic with respect to type. To highlight what changes with a non-monotonic cost, we first restate that result in the context of this study.

Corollary 1 *Assume that* $K''(\cdot) < 0$ *and that* $q^{*}(\theta) + K'(\theta) > 0$, $\forall\theta$. *First best is implemented if and only if either* (19) *and* (22) *are jointly satisfied, or* (19) *is violated and* (24) *is satisfied.*

When the cost increases with type everywhere the principal should adopt either lottery $\boldsymbol{\pi}^{\alpha}(\theta)$, which is most likely to satisfy limited liability constraints, or lottery $\boldsymbol{\pi}^{\gamma}(\theta)$, which is most likely to avoid conflicts between upward and downward incentive constraints. The former is preferable with a mild concavity of the opportunity cost, the latter with a more pronounced concavity. The choice between $\boldsymbol{\pi}^{\alpha}(\theta)$ and $\boldsymbol{\pi}^{\gamma}(\theta)$ explains the alternative conditions required in Corollary 1. The result in Proposition 2 is similar in this respect. Actually, the choice between lottery $\boldsymbol{\pi}^{\alpha}(\theta)$ and $\boldsymbol{\pi}^{\gamma}(\theta)$, and that between $\boldsymbol{\pi}^{\beta}(\theta)$ and $\boldsymbol{\pi}^{\gamma}(\theta)$, depend on the degree of concavity of $K(\cdot)$. This explains why in Proposition 2 there are two pairs of relevant conditions, namely (22) and (19) if $\theta < \widehat{\theta}$, and (23) and (20) if $\theta > \widehat{\theta}$.

There are nonetheless two essential differences between Proposition 2 and Corollary 1. In the framework of Proposition 2, $\boldsymbol{\pi}^{\gamma}(\theta)$ is more likely to be preferable to $\boldsymbol{\pi}^{\alpha}(\theta)$ and $\boldsymbol{\pi}^{\beta}(\theta)$ when the restrictions imposed by the non-monotonicity of the cost are severe (formally, when $h(\theta) \in \left(\underline{\theta}, \overline{\theta}\right)$), an issue which is of course absent in the context of Corollary 1. Less evident is that, if lottery $\boldsymbol{\pi}^{\gamma}(\theta)$ is adopted, then the relevant condition (24) is not equally tight when the cost is monotonic and when it is not. As stated in the next corollary, for some intermediate degree of concavity of $K(\cdot)$, the principal attains the first-best outcome (through lottery $\boldsymbol{\pi}^{\gamma}(\theta)$) when the cost increases for all types ((24) holds) but this is not necessarily the case otherwise (the proof is in Appendix G).

Corollary 2 *Assume that $K''(\cdot) < 0$.*

(i) Suppose that $q^*(\theta) + K'(\theta) > 0$, $\forall\theta$, and that $K(\cdot)$ is "sufficiently little concave" to have

$$\frac{q^*(\theta) + K'(x)}{q^*(\theta) + \frac{K(\theta)-K(x)}{\theta - x}} \leq \frac{\frac{p_1'(x)}{p_1(\theta)} - \frac{p_n'(x)}{p_n(\theta)}}{\frac{p_1(\theta)-p_1(x)}{(\theta-x)p_1(\theta)} - \frac{p_n(\theta)-p_n(x)}{(\theta-x)p_n(\theta)}}, \text{ if } x < \theta \tag{25}$$

$$\frac{q^*(\theta) + K'(x)}{q^*(\theta) + \frac{K(\theta)-K(x)}{\theta - x}} \geq \frac{\frac{p_1'(x)}{p_1(\theta)} - \frac{p_n'(x)}{p_n(\theta)}}{\frac{p_1(\theta)-p_1(x)}{(\theta-x)p_1(\theta)} - \frac{p_n(\theta)-p_n(x)}{(\theta-x)p_n(\theta)}}, \text{ if } x > \theta. \tag{26}$$

Then, (24) holds for all triplets $\left\{\theta^-, \theta, \theta^+\right\}$.

(ii) Suppose that $q^*(\theta) + K'(\theta) > 0$, $\forall\theta < \widehat{\theta}$, and $q^*(\theta) + K'(\theta) < 0$, $\forall\theta > \widehat{\theta}$, and that $K(\cdot)$ is "sufficiently little concave" that

1. *(25) holds if either $x > \theta$ and $x > h(\theta)$ or $x < \theta$ and $x < h(\theta)$;*

2. *(26) holds if $h(\theta) < x < \theta$ or $\theta < x < h(\theta)$.*

Then, (24) holds for all triplets $\left\{\theta^-, \theta, \theta^+\right\}$ if and only if

$$\frac{q^*(\theta) + K'(\theta)}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}} \leq L \sum_{s \neq 1,n} p_s(\theta) \left(\frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_s'(\theta)}{p_s(\theta)}}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}} - \frac{\frac{p_1(h(\theta))}{p_1(\theta)} - \frac{p_s(h(\theta))}{p_s(\theta)}}{\frac{p_1(h(\theta))}{p_1(\theta)} - \frac{p_n(h(\theta))}{p_n(\theta)}} \right), \text{ if } \theta < \widehat{\theta} \tag{27}$$

$$\frac{q^*(\theta) + K'(\theta)}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}} \geq L \sum_{s \neq 1,n} p_s(\theta) \left(\frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_s'(\theta)}{p_s(\theta)}}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}} - \frac{\frac{p_1(h(\theta))}{p_1(\theta)} - \frac{p_s(h(\theta))}{p_s(\theta)}}{\frac{p_1(h(\theta))}{p_1(\theta)} - \frac{p_n(h(\theta))}{p_n(\theta)}} \right), \text{ if } \theta > \widehat{\theta}. \tag{28}$$

The corollary presents results for the two cases of monotonic cost (case (i)) and non-monotonic cost (case (ii)). A common feature of the two cases is that the bonus in cost reimbursement, which the agent can appropriate by mimicking a type "distant" from the true type, is lower the less concave that the opportunity cost is. Thus, as long as the concavity of $K(\cdot)$ is sufficiently mild, correlated information is a powerful tool to make such a lie unprofitable. In either case, it follows that lotteries can be used to extract the associated benefit without, yet, attracting lies from other types which could rather gain from the lottery. However, by comparing the two cases, it emerges that the first-best allocation is at reach for different degrees of concavity of $K(\cdot)$. Whereas in case (i) it is not complicated to ensure that upward and downward incentive constraints are jointly satisfied, difficulties can arise in case (ii). To see this, first consider $\theta < \widehat{\theta}$ and suppose that the principal designs a lottery for type θ in such a way that there is no lower type θ^- which obtains any benefit, if it announces θ. Then, one cannot take for granted that such a lottery will also be unattractive to type θ^+. For instance, if $\theta^+ > \widehat{\theta}$, then, by reporting θ, type θ^+ might gain in both lottery and cost reimbursement, provided that the total cost decreases for types in $(\widehat{\theta}, \theta^+)$. In a similar fashion, when $\theta > \widehat{\theta}$ a lottery designed for type θ in such a way that it is unattractive to type θ^+, may end up attracting some type θ^- such that $\theta^- < \widehat{\theta} < \theta$. It follows that when the total cost is sharp sloping reverse U shaped, contractual efficiency is at reach for lower degrees of concavity of the opportunity cost than in the monotonic case.

Corollary 2 is also instructive about the level of liability which is required to attain efficiency when the total cost is reverse U shaped rather than monotonic. First consider the monotonic case and take (25) and (26) to be satisfied. Then, the required level of liability is pinned down by (22) and (23). Recall that these are the conditions under which the principal can construct a locally incentive compatible lottery which complies with the limited liability constraints. Global incentive compatibility (as expressed by (25) and (26)) is not related to limited liability, instead. Next consider a reverse U shaped cost and take (25) and (26) to be violated. Then, first-best implementation depends on whether or not it is possible to have (27) and (28) satisfied. As these conditions depend on L, it is apparent that, unlike with a monotonic cost, the possibility of attaining global incentive compatibility is now related to the magnitude of the deficits the agent can sustain. Moreover, global incentive compatibility (as now expressed by (27) and (28)) imposes more severe restrictions than local incentive compatibility, as the following corollary states (see Appendix H for the proof).

Corollary 3 *Condition* (27) *is tighter than* (22). *Condition* (28) *is tighter than* (23).

From Corollary 3 we deduce that a greater liability is required for the first-best allocation to be implemented with a sharp sloping reverse U shaped cost (case (ii) of Corollary 2) than with a monotonic cost (case (i) of Corollary 2). Provided in the former case it is more difficult to make "distant" types unattractive reports for the reasons previously explained, a greater risk exposure of the agent is necessary for the principal to be able to take enough advantage of the correlation between type and signal.

6 Discussion

We now clarify how our study is related to that of MRC. To that end, it is useful to recall that, in their setting without informative signals, the incentive constraint whereby θ is not an attractive report to any type θ' reduces to

$$\pi(\theta') \geq \pi(\theta) + (\theta - \theta') q(\theta) + K(\theta) - K(\theta'), \ \forall \theta' \neq \theta, \tag{29}$$

where, obviously, lotteries do not appear. There are essentially two cases to consider out of the analysis of MRC. We briefly review them below, discussing how useful non-monotonicity is to the principal when correlated information is available, relative to situations in which it is not.

1. $K''(\cdot) > -(q^*(\cdot))'$ From the analysis of MRC it emerges that, in this case, because the marginal cost with respect to type is different from zero ($q^*(\theta) + K'(\theta) \gtreqless 0$), the principal needs to concede information rents to (nearly) all types and distort quantities accordingly to contain those rents. This is in line with the classical result obtained by Byron and Myerson [3] in a model with monotonically increasing cost. However, unlike in that model, in MRC the incentives to cheat are not systematic across types in that the types below and above $\widehat{\theta} \in (\underline{\theta}, \overline{\theta})$ display opposite incentives. Furthermore, the incentives of all but the extreme types are strong when the countervailing effect induced by the opportunity cost is pronounced and, hence, the marginal opportunity cost is high. Being based on these results, it is difficult to conclude whether, in general, the second-best contractual attainments are closer to the first-best benchmark when the agent's cost is monotonic or, rather, when it is U shaped. In our setting with correlated information, we could establish that, regardless of whether the cost is monotonic or U shaped, the principal is able to design incentive compatible lotteries which extract all surplus from the agent, at least as long as his liability is not too little. That is, the principal implements first best in both situations, and one does not face the difficulty of assessing under what cost features the contractual attainments are closer to the respective first-best benchmarks.

2. $K''(\cdot) < -(q^*(\cdot))'$ In this case, information rents and output distortions reflect the fact that, unlike with a monotonically increasing cost, high types have incentives to understate information. MRC distinguish situations in which the marginal opportunity cost increases smoothly ($0 \leq K''(\cdot) < -(q^*(\cdot))'$) from situations in which it decreases with type ($K''(\cdot) < 0$). In the former case, incentives to cheat are generally weaker, and the principal is able to retain all surplus from a bunch of intermediate types, which display the weakest incentives. By contrast, in the latter case, all types but $\widehat{\theta}$ obtain an information rent. This suggests that contracting is more efficient when the opportunity cost is slightly convex rather than concave. Less clear is whether the principal is better off when the agent's cost is reverse U shaped rather than monotonic. Assuming that $K(\cdot)$ is concave, Lewis and Sappington [14] demonstrate that countervailing incentives enhance contracting. However, that result is derived in a specific

example and it is difficult to assess its generality. Our analytical framework enables us to be more conclusive, in that respect, with regards to environments with *ex-post* informative signals. We establish that the countervailing effect on incentives associated with the opportunity cost, when this is concave, imposes contractual restrictions and, unlike with different shapes of $K(\cdot)$, first-best implementation may not be at reach. We can thus conclude that when the total cost is reverse U shaped and sharp sloping ($K(\cdot)$ is concave), in our framework, non-monotonicity is unfavorable to the principal, at odds with the conclusion Lewis and Sappington [14] draw in their example without signals.

7 Conclusion

We investigated first-best implementation in a principal-agent model with correlated information in which the agent's cost is non-monotonic with respect to type, as in the literature on countervailing incentives. We showed that constructing incentive compatible lotteries under limited liability is a trickier task as compared to situations in which the cost is monotonic, instead. Moreover, when the cost is reverse U shaped first best is implemented under more restrictive conditions than with a monotonic cost. Interestingly, this is at odds with settings without informative signals, in which, as Lewis and Sappington [14] - [15] suggest, the principal may want to create or reinforce countervailing incentives to enhance contracting.

We highlighted how the features of the optimal lottery are tied, on the one hand, to the specific type of the agent and, on the other, to the shape of the opportunity cost faced by the agent in the trade with the principal. Unlike in settings without correlated information, in our framework the principal's contractual attainment (namely, whether or not she implements first best) only depends on how concave the opportunity cost is with respect to type, given the agent's liability. However, like in those settings, whether the opportunity cost is highly convex, slightly convex, or concave is critical to the optimal contractual design. Considering the large variety of real-world agency relationships characterized by non-monotonicities of the kind represented in our model, it is important to gain a full understanding of the optimal lottery design for practical use in those environments when correlated information is available.

Whereas we focused on first-best implementation, a natural observation would be that a full-fledged analysis would require looking at second-best contractual design as well. In fact, this exercise might be lengthy and tedious but not particularly insightful. On the one hand, the standard first-order approach is unlikely to be applicable with a continuum of types in that the contractual allocation may not be differentiable, a concern already expressed in previous studies (see Kessler *et al.* [12], for instance). On the other hand, even if the technical complications are addressed, one would expect the second-best solution to display similar features to the solution characterized by MRC, up to some enhancements in the efficiency of the contractual allocation as induced by the introduction of lotteries and depending on the intensity of the correlation between signal and type. This all motivated us to restrict attention to first-best implementation.

References

[1] Acconcia, A., R. Martina and S. Piccolo (2008), "Vertical Restraints under Asymmetric Information: On the Role of Participation Constraints," *Journal of Industrial Economics*, 56(2), 379-401

[2] Aguirre, I., and A. Beitia (2008), "Regulating a Multiproduct Monopolist with Unknown Demand: Cross-Subsidization and Countervailing Incentives," *Journal of Theoretical and Institutional Economics*, 164(4), 652-675

[3] Baron, D.P., and R.B. Myerson (1982), "Regulating a Monopolist with Unknown Costs," *Econometrica*, 50(4), 911-930

[4] Boone, J., and C. Schottmuller (2016), "Procurement with specialized firms," *The RAND Journal of Economics*, 47(3), 661-687

[5] Che, L.Y., and D.E.M. Sappington (2010), "Contracting with private knowledge of signal quality," *The RAND Journal of Economics*, 41(2), 244-269

[6] Crémer, J., and R. McLean (1988), "Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions," *Econometrica*, 56(6), 1247-1258

[7] Danau, D., and A. Vinella (2016), "On the optimal use of correlated information in contractual design under limited liability," *CREM Working Papers*, No. 2016-05

[8] Degryse, H., and A. de Jong (2006), "Investment and Internal Finance: Asymmetric Information or Managerial Discretion", *International Journal of Industrial Organization*, 24, 125-147

[9] Demougin, D.M., and D.A. Garvie (1991), "Contractual design with correlated information under limited liability," *The RAND Journal of Economics*, 22(4), 477-489

[10] Gary-Bobo, R., and Y. Spiegel (2006), "Optimal State-Contingent Regulation under Limited Liability," *RAND Journal of Economics*, 37, 431-448

[11] Jullien, B. (2000), "Participation Constraints in Adverse Selection Models," *Journal of Economic Theory*, 93, 1-47

[12] Kessler, A.S., C. Lulfesmann and P.W. Schmitz (2005), "Endogenous punishments in agency with verifiable ex post information," *International Economic Review*, 46(4), 1207-1231

[13] Krause, A. (2009), "Education and Taxation Policies in the Presence of Countervailing Incentives," *Economica*, 76, 387-399

[14] Lewis, T.R., and D.E.M. Sappington (1989a), "Countervailing Incentives in Agency Problems," *Journal of Economic Theory*, 49, 294-313

[15] Lewis, T.R., and D.E.M. Sappington (1989b), "Inflexible Rules in Incentive Problems," *American Economic Review*, 79(1), 69-84

[16] Maggi, G., and A. Rodriguez-Clare (1995), "On Countervailing Incentives," *Journal of Economic Theory*, 66, 238-263

[17] Ollier, S., and L. Thomas (2013), "Ex post participation constraint in a principal-agent model with adverse selection and moral hazard," *Journal of Economic Theory*, 148, 2383-2403

[18] Poudou, J.C., M. Roland and L. Thomas (2009), "Universal Service Obligations and Competition with Asymmetric Information," *The B.E. Journal of Theoretical Economics*, 9(1) (Topics), Article 35

[19] Riordan, M., and D.E.M. Sappington (1988), "Optimal Contracts with Public Ex Post Information," *Journal of Economic Theory*, 45, 189-199

A The lottery design

A.1 Derivation of (6) and (7)

Using (2), we rewrite (5) as

$$(\theta-\theta')\, q^{*}(\theta)+K(\theta)-K(\theta') \leq \sum_{s=1}^{n} \pi_s(\theta)(p_s(\theta)-p_s(\theta')). \tag{30}$$

From (2) we also obtain the following expression of $\pi_1(\theta)$:

$$\pi_1(\theta) = -\sum_{s=2}^{n} \pi_s(\theta) \frac{p_s(\theta)}{p_1(\theta)}.$$

Replacing in (30) we obtain a formulation without $\pi_1(\theta)$, namely:

$$\begin{aligned} & (\theta-\theta')\, q^{*}(\theta)+K(\theta)-K(\theta') \\ \leq\ & \sum_{s\neq 1,n} \pi_s(\theta)\, p_s(\theta) \left(\frac{p_1(\theta')}{p_1(\theta)}-\frac{p_s(\theta')}{p_s(\theta)}\right)+\pi_n(\theta)\, p_n(\theta) \left(\frac{p_1(\theta')}{p_1(\theta)}-\frac{p_n(\theta')}{p_n(\theta)}\right), \end{aligned}$$

which is equivalent to

$$\begin{aligned} & \pi_n(\theta)\, p_n(\theta) \left(\frac{p_1(\theta')}{p_1(\theta)}-\frac{p_n(\theta')}{p_n(\theta)}\right) \\ \geq\ & (\theta-\theta')\, q^{*}(\theta)+K(\theta)-K(\theta')-\sum_{s\neq 1,n} \pi_s(\theta)\, p_s(\theta) \left(\frac{p_1(\theta')}{p_1(\theta)}-\frac{p_s(\theta')}{p_s(\theta)}\right). \end{aligned} \tag{31}$$

First take $\theta'=\theta^{-}<\theta$. Then, dividing both sides of (31) by $p_n(\theta)\left(\frac{p_n(\theta^{-})}{p_n(\theta)}-\frac{p_1(\theta^{-})}{p_1(\theta)}\right)$, which is positive given (3), we obtain (6). Next take $\theta'=\theta^{+}>\theta$. Then, dividing both sides of (31) by $p_n(\theta)\left(\frac{p_1(\theta^{+})}{p_1(\theta)}-\frac{p_n(\theta^{+})}{p_n(\theta)}\right)$, which is positive given (3), we obtain (7).

A.2 Derivation of (12) and (13)

Using $\pi_s(\theta)=\pi_n(\theta)$, $\forall s\neq 1,n$, we rewrite (8) as

$$\pi_n(\theta) = \frac{q^{*}(\theta)+K'(\theta)+\pi_n(\theta)\sum_{s\neq 1,n} p_s(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_s'(\theta)}{p_s(\theta)}\right)}{-p_n(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_n'(\theta)}{p_n(\theta)}\right)},$$

which further becomes

$$-\pi_n(\theta)\, p_n(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_n'(\theta)}{p_n(\theta)}\right) = q^{*}(\theta)+K'(\theta)+\pi_n(\theta)\sum_{s\neq 1,n} p_s(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_s'(\theta)}{p_s(\theta)}\right).$$

Regrouping terms with $\pi_n(\theta)$ yields

$$\begin{aligned}
-\left(q^*(\theta)+K'(\theta)\right) &= \pi_n(\theta)\left[p_n(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_n'(\theta)}{p_n(\theta)}\right)+\sum_{s\neq 1,n}p_s(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_s'(\theta)}{p_s(\theta)}\right)\right] \\
&= \pi_n(\theta)\sum_{s\neq 1}p_s(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_s'(\theta)}{p_s(\theta)}\right) \\
&= \pi_n(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)}\sum_{s\neq 1}p_s(\theta)-\sum_{s\neq 1}p_s'(\theta)\right).
\end{aligned}$$

Using $\sum_{s\neq 1}p_s(\theta)=1-p_1(\theta)$ and $\sum_{s\neq 1}p_s'(\theta)=-p_1'(\theta)$, we rewrite

$$\begin{aligned}
-\left(q^*(\theta)+K'(\theta)\right) &= \pi_n(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)}-p_1'(\theta)+p_1'(\theta)\right) \\
&= \pi_n(\theta)\frac{p_1'(\theta)}{p_1(\theta)}.
\end{aligned}$$

Rearranging and recalling that $\pi_s(\theta)=\pi_n(\theta)\,,\ \forall s\neq 1,n$, (13) is obtained.

Replacing (13) in $\pi_1(\theta)=-\sum_{s\neq 1}\pi_s(\theta)\frac{p_s(\theta)}{p_1(\theta)}$ yields

$$\pi_1(\theta)=\left(q^*(\theta)+K'(\theta)\right)\sum_{s\neq 1}\frac{p_s(\theta)}{p_1'(\theta)}.$$

Because $\sum_{s\neq 1}p_s(\theta)=1-p_1(\theta)$, (12) is obtained.

A.3 Derivation of (14) and (15)

The procedure is similar to that followed to derive (12) and (13), except that (2) is now used to obtain an expression of $\pi_n(\theta)$ (rather than of $\pi_1(\theta)$) and the local incentive constraint is rewritten in terms of $\pi_1(\theta)$ (rather than of $\pi_n(\theta)$, as in (8)). Specifically, from (2) we have

$$\pi_n(\theta)=-\pi_1(\theta)\frac{p_1(\theta)}{p_n(\theta)}-\sum_{s\neq 1,n}\pi_s(\theta)\frac{p_s(\theta)}{p_n(\theta)}. \tag{32}$$

We use the latter to rewrite (5) for θ^- and θ^+, as follows:

$$\pi_1(\theta) \geq \frac{\left(\theta-\theta^-\right)\left(q^*(\theta)+\frac{K(\theta)-K(\theta^-)}{\theta-\theta^-}\right)-\sum_{s\neq 1,n}\pi_s(\theta)\,p_s(\theta)\left(\frac{p_n(\theta^-)}{p_n(\theta)}-\frac{p_s(\theta^-)}{p_s(\theta)}\right)}{p_1(\theta)\left(\frac{p_n(\theta^-)}{p_n(\theta)}-\frac{p_1(\theta^-)}{p_1(\theta)}\right)} \tag{33}$$

$$\pi_1(\theta) \leq \frac{\left(\theta^+-\theta\right)\left(q^*(\theta)+\frac{K(\theta)-K(\theta^+)}{\theta-\theta^+}\right)+\sum_{s\neq 1,n}\pi_s(\theta)\,p_s(\theta)\left(\frac{p_n(\theta^+)}{p_n(\theta)}-\frac{p_s(\theta^+)}{p_s(\theta)}\right)}{p_1(\theta)\left(\frac{p_1(\theta^+)}{p_1(\theta)}-\frac{p_n(\theta^+)}{p_n(\theta)}\right)}. \tag{34}$$

Setting $\pi_1(\theta) = \pi_s(\theta)$, $\forall s \neq 1, n$, in these inequalities yields:

$$\pi_1(\theta) \geq \frac{(\theta - \theta^-)\, p_n(\theta)}{p_n(\theta^-) - p_n(\theta)} \left(q^*(\theta) + \frac{K(\theta) - K(\theta^-)}{\theta - \theta^-} \right) \tag{35}$$

$$\pi_1(\theta) \leq \frac{(\theta^+ - \theta)\, p_n(\theta)}{p_n(\theta) - p_n(\theta^+)} \left(q^*(\theta) + \frac{K(\theta) - K(\theta^+)}{\theta - \theta^+} \right). \tag{36}$$

As $\theta^- \to \theta$ and $\theta^+ \to \theta$ these conditions are jointly satisfied if and only if $\pi_1(\theta)$ is given by:

$$\pi_1(\theta) = (q^*(\theta) + K'(\theta)) \frac{p_n(\theta)}{-p'_n(\theta)}.$$

Because $\pi_1(\theta) = \pi_s(\theta)$, $\forall s \neq 1, n$, (15) is obtained. Using this in (32), (14) is obtained.

A.4 Derivation of (16) and (17)

Set $\pi_s(\theta) = -L$ in (8) to get

$$\begin{aligned}
\pi_n(\theta) &= \frac{q^*(\theta) + K'(\theta) - L \sum_{s\neq 1,n} p_s(\theta) \left(\frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_s(\theta)}{p_s(\theta)} \right)}{-p_n(\theta) \left(\frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)} \right)} \\
&= \frac{q^*(\theta) + K'(\theta) - L \left(\frac{p'_1(\theta)}{p_1(\theta)} \sum_{s\neq 1,n} p_s(\theta) - \sum_{s\neq 1,n} p'_s(\theta) \right)}{-p_n(\theta) \left(\frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)} \right)}.
\end{aligned}$$

Using $\sum_{s\neq 1,n} p_s(\theta) = 1 - p_1(\theta) - p_n(\theta)$ and $\sum_{s\neq 1,n} p'_s(\theta) = -p'_1(\theta) - p'_n(\theta)$ further yields

$$\pi_n(\theta) = \frac{q^*(\theta) + K'(\theta) - L\frac{p'_1(\theta)}{p_1(\theta)} + Lp_n(\theta) \left(\frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)} \right)}{-p_n(\theta) \left(\frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)} \right)},$$

from which (17) is obtained.

Replace (17), together with $\pi_s(\theta) = -L$, $\forall s \neq 1, n$, in $\pi_1(\theta) = -\sum_{s\neq 1} \pi_s(\theta) \frac{p_s(\theta)}{p_1(\theta)}$. It yields

$$\begin{aligned}
\pi_1(\theta) &= L \sum_{s\neq 1,n} \frac{p_s(\theta)}{p_1(\theta)} - \left[\frac{L\frac{p'_1(\theta)}{p_1(\theta)} - (q^*(\theta) + K'(\theta))}{p_n(\theta) \left(\frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)} \right)} - L \right] \frac{p_n(\theta)}{p_1(\theta)} \\
&= L \sum_{s\neq 1,n} \frac{p_s(\theta)}{p_1(\theta)} - \frac{L\frac{p'_1(\theta)}{p_1(\theta)} - (q^*(\theta) + K'(\theta))}{p_n(\theta) \left(\frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)} \right)} \frac{p_n(\theta)}{p_1(\theta)} + L\frac{p_n(\theta)}{p_1(\theta)} \\
&= L \sum_{s\neq 1} \frac{p_s(\theta)}{p_1(\theta)} - \frac{L\frac{p'_1(\theta)}{p_1(\theta)} - (q^*(\theta) + K'(\theta))}{p_1(\theta) \left(\frac{p'_1(\theta)}{p_1(\theta)} - \frac{p'_n(\theta)}{p_n(\theta)} \right)}.
\end{aligned}$$

Using $\sum_{s\neq 1,n} p_s(\theta) = 1 - p_1(\theta)$ and $\sum_{s\neq 1,n} p_s'(\theta) = -p_1'(\theta)$,this further becomes

$$\begin{aligned}
\pi_1(\theta) &= \frac{q^*(\theta) + K'(\theta) - L\frac{p_1'(\theta)}{p_1(\theta)}}{p_1(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right)} + L\frac{1-p_1(\theta)}{p_1(\theta)} \\
&= \frac{q^*(\theta) + K'(\theta) - L\frac{p_1'(\theta)}{p_1(\theta)} + L\left[\frac{p_1'(\theta)}{p_1(\theta)} - p_1'(\theta) - \frac{p_n'(\theta)}{p_n(\theta)}(1-p_1(\theta))\right]}{p_1(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right)} \\
&= \frac{q^*(\theta) + K'(\theta) - L\frac{p_n'(\theta)}{p_n(\theta)} - L\left(p_1'(\theta) - p_1(\theta)\frac{p_n'(\theta)}{p_n(\theta)}\right)}{p_1(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right)},
\end{aligned}$$

from which (16) is derived.

B Proof of Lemma 3

Here and in subsequent proofs, we will be based on the equivalence between $K''(\cdot) < 0$ and the following:

$$K'(x) > \frac{K(\theta) - K(x)}{\theta - x} > K'(\theta), \ \forall x < \theta \tag{37}$$

$$K'(x) < \frac{K(x) - K(\theta)}{x - \theta} < K'(\theta), \ \forall x > \theta. \tag{38}$$

(I) $\theta < \widehat{\theta}$. Replacing x with $h(\theta)$ in (37) and (38), we can write

$$K''(\cdot) < 0 \Leftrightarrow \begin{cases} \frac{K(h(\theta)) - K(\theta)}{h(\theta) - \theta} > K'(\theta) \text{ if } h(\theta) < \theta \\ \\ \frac{K(h(\theta)) - K(\theta)}{h(\theta) - \theta} < K'(\theta) \text{ if } h(\theta) > \theta \end{cases}.$$

Adding $q(\theta)$ on each side of the right-hand conditions above and using the definition of $h(\theta)$ in (21), we further deduce the following:

$$K''(\cdot) < 0 \Leftrightarrow \begin{cases} q^*(\theta) + K'(\theta) < 0 \text{ if } h(\theta) < \theta \\ \\ q^*(\theta) + K'(\theta) > 0 \text{ if } h(\theta) > \theta \end{cases} \tag{39}$$

Recall that for $\theta < \widehat{\theta}$ it is $q^*(\theta) + K'(\theta) < 0$ if and only if $K''(\cdot) \geq -(q^*(\theta))'$. We use this in (39) to deduce the following:

- if $K''(\cdot) < 0$, then $q^*(\theta) + K'(\theta) > 0$, and hence $h(\theta) > \theta$;
- if $0 \leq K''(\cdot) < -(q^*(\cdot))'$, then $q^*(\theta) + K'(\theta) > 0$, and hence $h(\theta) < \theta$;
- if $K''(\cdot) > -(q^*(\cdot))'$, then $q^*(\theta) + K'(\theta) < 0$, and hence $h(\theta) > \theta$.

(II) $\theta > \widehat{\theta}$. Recall that for $\theta > \widehat{\theta}$ it is $q^*(\theta) + K'(\theta) < 0$ if and only if $K''(\cdot) < -(q^*(\theta))'$. We use this in (39) to deduce the following:

- if $K''(\cdot) < 0$, then $q^*(\theta) + K'(\theta) < 0$, and hence $h(\theta) < \theta$;
- if $0 \leq K''(\cdot) < -(q^*(\cdot))'$, then $q^*(\theta) + K'(\theta) < 0$, and hence $h(\theta) > \theta$;
- if $K''(\cdot) \geq -(q^*(\cdot))'$, then $q^*(\theta) + K'(\theta) > 0$, and hence $h(\theta) < \theta$.

C Proof of Lemma 4

(I) $0 \leq K''(\cdot) < -(q^*(\cdot))'$

$(I.1)$ $\theta < \widehat{\theta}$, in which case $q^*(\theta) + K'(\theta) > 0$. Recall that $\left(IC^{\theta}_{\theta^-}\right)$ and $\left(IC^{\theta}_{\theta^+}\right)$ are rewritten as (6) and (7). Setting $\pi_n(\theta) = \pi_s(\theta)$ in (6) and (7), they are further rewritten as

$$\pi_n(\theta) \leq \frac{-\left(\theta-\theta^-\right)p_1(\theta)}{p_1(\theta)-p_1(\theta^-)}\left(q^*(\theta)+\frac{K(\theta)-K\left(\theta^-\right)}{\theta-\theta^-}\right) \tag{40}$$

$$\pi_n(\theta) \geq \frac{-\left(\theta^+-\theta\right)p_1(\theta)}{p_1(\theta^+)-p_1(\theta)}\left(q^*(\theta)+\frac{K\left(\theta^+\right)-K(\theta)}{\theta^+-\theta}\right). \tag{41}$$

$\exists \pi_n(\theta)$ such that both of these conditions hold if and only if:

$$\begin{aligned} &\frac{\theta-\theta^-}{p_1(\theta)-p_1(\theta^-)}\left(q^*(\theta)+\frac{K(\theta)-K\left(\theta^-\right)}{\theta-\theta^-}\right) \\ \leq\; &\frac{\theta^+-\theta}{p_1(\theta^+)-p_1(\theta)}\left(q^*(\theta)+\frac{K\left(\theta^+\right)-K(\theta)}{\theta^+-\theta}\right), \end{aligned} \tag{42}$$

which is (11) as rewritten for $\pi_n(\theta) = \pi_s(\theta)$**.** From (38), we have $\frac{K\left(\theta^+\right)-K(\theta)}{\theta^+-\theta} > K'(\theta)$. Because $q^*(\theta)+K'(\theta) > 0$, we also have $q^*(\theta)+\frac{K\left(\theta^+\right)-K(\theta)}{\theta^+-\theta} > 0$ so that (42) is rewritten as (19). Because $p_1''(\cdot) < 0$ (under (4)), the right-hand side of (19) is above one. Because $K''(\cdot) > 0$ implies $\frac{K\left(\theta^+\right)-K(\theta)}{\theta^+-\theta} > \frac{K(\theta)-K\left(\theta^-\right)}{\theta-\theta^-}$, the left-hand side of (19) is below one. Hence, in this case, (19) is satisfied.

$(I.2)$ $\theta > \widehat{\theta}$, in which case $q^*(\theta) + K'(\theta) < 0$. Recall that setting $\pi_1(\theta) = \pi_s(\theta)$, $\left(IC^{\theta}_{\theta^-}\right)$ and $\left(IC^{\theta}_{\theta^+}\right)$ are rewritten as (35) and (36). $\exists \pi_1(\theta)$ such that both conditions hold if and only if:

$$\begin{aligned} &\frac{\theta-\theta^-}{p_n(\theta^-)-p_n(\theta)}\left(q^*(\theta)+\frac{K(\theta)-K\left(\theta^-\right)}{\theta-\theta^-}\right) \\ \leq\; &\frac{\theta^+-\theta}{p_n(\theta)-p_n(\theta^+)}\left(q^*(\theta)+\frac{K\left(\theta^+\right)-K(\theta)}{\theta^+-\theta}\right), \end{aligned} \tag{43}$$

which is (11) rewritten for $\pi_1(\theta) = \pi_s(\theta)$. From (37) and (38), we have $\frac{K\left(\theta^+\right)-K(\theta)}{\theta^+-\theta} > K'(\theta) > \frac{K(\theta)-K\left(\theta^-\right)}{\theta-\theta^-}$. Because $q^*(\theta) + K'(\theta) < 0$, we also have $q^*(\theta) + \frac{K(\theta)-K\left(\theta^-\right)}{\theta-\theta^-} < 0$. (43) is satisfied when $q^*(\theta) + \frac{K\left(\theta^+\right)-K(\theta)}{\theta^+-\theta} > 0 \Leftrightarrow \theta^+ > h(\theta)$, which is true because $h(\theta) < \widehat{\theta} < \theta$ and $\theta < \theta^+$.

(II) $K''(\cdot) \geq -(q^*(\cdot))'$

$(II.1)$ $\theta < \widehat{\widehat{\theta}}$, in which case $q^*(\theta) + K'(\theta) < 0$. Recall from Lemma 3 that $\theta < h(\theta)$. If $\theta^+ < h(\theta)$, then the right-hand side of (43) is negative. The left-hand side of (43) is negative as well. Hence, (43) is rewritten as (20). Because $p_n''(\cdot) < 0$ given (4), the left-hand side of (20) is below 1. Moreover, because $K''(\cdot) > 0$, the right-hand side of (20) is above 1. Hence, (20) is satisfied.

$(II.2)$ $\theta > \widehat{\theta}$, in which case $q^*(\theta) + K'(\theta) > 0$. Because $q^*(\theta) + \frac{K(\theta^+)-K(\theta)}{\theta^+-\theta} > 0$, (42) is rewritten as (19). Because $p_1''(\cdot) < 0$ (under (4)), the right-hand side of (19) is higher than 1. Because $K''(\cdot) > 0$, the left-hand side of (19) is lower than 1. Hence, (19) holds.

D Proof of Proposition 1

For $\theta < \widehat{\theta}$ the optimal lottery is $\boldsymbol{\pi}^{\alpha}(\theta)$ (Lemma 4). (LL_{θ}^{s}) is satisfied for all types if and only if $\pi_s(\theta) \geq -L$, $\forall\theta$, $\forall s \neq 1$. Using (13), this inequality is rewritten as (22).

Analogously, for $\theta > \widehat{\theta}$ the optimal lottery is $\boldsymbol{\pi}^{\beta}(\theta)$ (Lemma 4). (LL_{θ}^{s}) is satisfied for all types if and only if $\pi_s(\theta) \geq -L$, $\forall\theta$, $\forall s \neq n$. Using (15), this inequality is rewritten as (23).

E Proof of Lemma 5

(I) $\theta < \widehat{\theta}$. In this case, $q^*(\theta) + K'(\theta) > 0$ and the choice is between $\boldsymbol{\pi}^{\alpha}(\theta)$ and $\boldsymbol{\pi}^{\gamma}(\theta)$ (Lemma 1 and 2). Because $K''(\cdot) < 0$ it follows from (37) that $\frac{K(\theta)-K(\theta^-)}{\theta-\theta^-} > K'(\theta)$, and from (38) that $K'(\theta) > \frac{K(\theta^+)-K(\theta)}{\theta^+-\theta}$. Moreover, because $q^*(\theta) + K'(\theta) > 0$, we have $q^*(\theta) + \frac{K(\theta)-K(\theta^-)}{\theta-\theta^-} > 0$. If $\theta^+ < h(\theta)$, then:

$$q^*(\theta) + \frac{K(\theta) - K(h(\theta))}{\theta - h(\theta)} = 0 < q^*(\theta) + \frac{K(\theta) - K(\theta^+)}{\theta - \theta^+}$$

and (42) is rewritten as (19). Both the left-hand side and the right-hand side of (19) are positive so that (19) is not implied by the assumptions of the model and is to be verified. If (19) is satisfied, and hence (11) is satisfied with $\boldsymbol{\pi}^{\alpha}(\theta)$, then $\boldsymbol{\pi}^{\alpha}(\theta)$ is optimal (Lemma 1 and 2). If $\theta^+ \geq h(\theta)$, then with analogous reasoning we deduce that $q^*(\theta) + \frac{K(\theta)-K(\theta^+)}{\theta-\theta^+} < 0$ so that the right-hand side of (42) is negative. Because the left-hand side is positive, (42) is violated. Hence, (11) is not satisfied with $\boldsymbol{\pi}^{\alpha}(\theta)$ and $\boldsymbol{\pi}^{\gamma}(\theta)$ is optimal.

(II) $\theta > \widehat{\theta}$. In this case, $q^*(\theta) + K'(\theta) < 0$ and the choice is between $\boldsymbol{\pi}^{\beta}(\theta)$ and $\boldsymbol{\pi}^{\gamma}(\theta)$ (Lemma 1 and 2). Because $K''(\cdot) < 0$, it follows from (38) that $\frac{K(\theta^+)-K(\theta)}{\theta^+-\theta} < K'(\theta)$, and from (37) that $K'(\theta) < \frac{K(\theta)-K(\theta^-)}{\theta-\theta^-}$. Moreover, because $q^*(\theta) + K'(\theta) < 0$, we have $q^*(\theta) + \frac{K(\theta^+)-K(\theta)}{\theta^+-\theta} < 0$. If $\theta^- > h(\theta)$, then:

$$q^*(\theta) + \frac{K(\theta) - K(\theta^-)}{\theta - \theta^-} < q^*(\theta) + \frac{K(\theta) - K(h(\theta))}{\theta - h(\theta)} = 0$$

and (43) is rewritten as (20). With $p_n''(\cdot) < 0$ (under (4)), the left-hand side of (20) is below 1. Because $\frac{K(\theta)-K(\theta^-)}{\theta-\theta^-} > \frac{K(\theta^+)-K(\theta)}{\theta^+-\theta}$, the right-hand side of (20) is below 1 as well. Hence, we cannot conclude that (20) is satisfied and it must be verified. If (20) is satisfied, and hence (11) is satisfied with $\boldsymbol{\pi}^{\beta}(\theta)$, then $\boldsymbol{\pi}^{\beta}(\theta)$ is optimal (Lemma 1 and 2). If $\theta^- \leq h(\theta)$, then with analogous reasoning we deduce that $q^*(\theta) + \frac{K(\theta)-K(\theta^-)}{\theta-\theta^-} > 0$, involving that the left-hand side of (43) is positive and the condition is violated. Hence, (11) is not satisfied with $\boldsymbol{\pi}^{\beta}(\theta)$ and $\boldsymbol{\pi}^{\gamma}(\theta)$ is optimal.

F Proof of Proposition 2

Derivation of (24)

Given (3), the difference in the brackets multiplied by $\pi_s(\theta)\,p_s(\theta)$ in (11) is negative for all triplets $\left\{\theta^-,\theta,\theta^+\right\}$. Hence, (11) is weakest when $\pi_s(\theta)=-L$, $\forall s\neq 1,n$. Replacing these values, (11) is reformulated as (24).

Limited liability constraints are satisfied

We are left with verifying (LL_θ^1) and (LL_θ^n).

(I) $\theta<\widehat{\theta}$. From (17), we see that (LL_θ^n) holds because $\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_n'(\theta)}{p_n(\theta)}>0$ (given (3)) and $q^*(\theta)+K'(\theta)\leq Lp_1'(\theta)/p_1(\theta)$ (by (22)). From (16), we see that (LL_θ^1) holds because $\frac{p_1'(\theta)}{p_1(\theta)}-\frac{p_n'(\theta)}{p_n(\theta)}>0$ and $p_n'(\theta)<0$ (given (3)) together with $q^*(\theta)+K'(\theta)>0$.

(II) $\theta>\widehat{\theta}$. From (16), we see that (LL_θ^1) holds because $p_n'(\theta)<0$ (given (3)) and $q^*(\theta)+K'(\theta)\geq Lp_n'(\theta)/p_n(\theta)$ (by (23)). From (17), we see that (LL_θ^n) holds because $q^*(\theta)+K'(\theta)<0$.

Overall, (LL_θ^1) and (LL_θ^n) are satisfied $\forall\theta$.

G Proof of Corollary 2

Define the functions:

$$\varphi(x)\equiv\frac{q^*(\theta)+\frac{K(\theta)-K(x)}{\theta-x}}{\frac{p_1(\theta)-p_1(x)}{(\theta-x)p_1(\theta)}-\frac{p_n(\theta)-p_n(x)}{(\theta-x)p_n(\theta)}}$$

and $g(\theta)$ such that $q^*(\theta)+K'(g(\theta))=0$. We will identify the conditions under which $\varphi'(x)\geq 0$ for $x=\theta^-$ and for $x=\theta^+$. We have $\varphi'(x)\geq 0$ if and only if:

$$\begin{aligned}&(q^*(\theta)+K'(x))\left(\frac{p_1(\theta)-p_1(x)}{p_1(\theta)}-\frac{p_n(\theta)-p_n(x)}{p_n(\theta)}\right)\\ \leq\;&(\theta-x)\left(q^*(\theta)+\frac{K(\theta)-K(x)}{\theta-x}\right)\left(\frac{p_1'(x)}{p_1(\theta)}-\frac{p_n'(x)}{p_n(\theta)}\right).\end{aligned}\tag{44}$$

Under (3), $\frac{p_1(\theta)-p_1(x)}{p_1(\theta)}-\frac{p_n(\theta)-p_n(x)}{p_n(\theta)}>0$ if and only if $x<\theta$. Hence, (44) is equivalent to the following pair of conditions:

$$q^*(\theta)+K'(x)\;\leq\;\left(q^*(\theta)+\frac{K(\theta)-K(x)}{\theta-x}\right)\frac{\frac{p_1'(x)}{p_1(\theta)}-\frac{p_n'(x)}{p_n(\theta)}}{\frac{p_1(\theta)-p_1(x)}{(\theta-x)p_1(\theta)}-\frac{p_n(\theta)-p_n(x)}{(\theta-x)p_n(\theta)}},\ \text{if } x<\theta\tag{45}$$

$$q^*(\theta)+K'(x)\;\geq\;\left(q^*(\theta)+\frac{K(\theta)-K(x)}{\theta-x}\right)\frac{\frac{p_1'(x)}{p_1(\theta)}-\frac{p_n'(x)}{p_n(\theta)}}{\frac{p_1(\theta)-p_1(x)}{(\theta-x)p_1(\theta)}-\frac{p_n(\theta)-p_n(x)}{(\theta-x)p_n(\theta)}},\ \text{if } x>\theta.\tag{46}$$

(I) either $\nexists\widehat{\theta}$ or $\widehat{\theta}>\overline{\theta}$. In this case, $q^*(\theta)+K'(x)>0$ and $q^*(\theta)+\frac{K(\theta)-K(x)}{\theta-x}>0$, $\forall x$, $\forall\theta$. For $x<\theta$ (44) is equivalent to (45), which is reformulated as (25); for $x>\theta$ (44) is equivalent to (46), which is reformulated as (26). If $K(\cdot)$ is sufficiently little concave that (25) and (26) hold, then the left-hand side of (24) increases with θ^- and decreases with θ^+. As the right-hand side of (24) decreases with θ^- and increases with θ^+ (given (3)), it follows that (24) is tightest as $\theta^-\to\theta$ and $\theta^+\to\theta$, in which case it is satisfied. Hence, (24) is satisfied any triplet $\left\{\theta^-,\theta,\theta^+\right\}$.

(II) $\widehat{\theta}\in\left(\underline{\theta},\overline{\theta}\right)$. Using $\frac{d}{dx}\left(\frac{K(x)-K(\theta)}{x-\theta}\right)<0$ (which follows from $K''(\cdot)<0$) together with the definition of $h(\theta)$, we see that $q^*(\theta) + \frac{K(\theta)-K(x)}{\theta - x} > 0$ if and only if $x < h(\theta)$. (44) is rewritten as (25), if either $x < \theta$ and $x < h(\theta)$ or $x > \theta$ and $x > h(\theta)$; it is rewritten as (26), if either $x < \theta$ and $x > h(\theta)$ or $x > \theta$ and $x < h(\theta)$.

We now verify the pairs $\{\theta, x\}$ for which the associated condition (25) or (26) is violated. Because $K''(\cdot) < 0$, $\widehat{\theta} < g(\theta) < h(\theta)\ \forall \theta < \widehat{\theta}$ and $h(\theta) < g(\theta) < \widehat{\theta}\ \forall \theta > \widehat{\theta}$.

$(II.1)$ If $\theta < \widehat{\theta}$ together with $g(\theta) < x < h(\theta)$, then the associated condition (26) is violated and $\varphi'(x) < 0$ for any degree of concavity of $K(\cdot)$. If $x \notin [g(\theta), h(\theta)]$, then the associated condition (25) or (26) can be satisfied. Because $\theta^- < \theta < g(\theta)$, only θ^+ can belong to $[g(\theta), h(\theta)]$. Hence, for any $\theta^- < \theta$, $\varphi'\left(\theta^-\right) > 0$ if the associated condition (25) or (26) is satisfied so that (24) is tightest as $\theta^- \to \theta$. Therefore, (24) is to be verified for $\theta^- \to \theta$ and some $\theta^+ > \theta$ (see below).

$(II.2)$ If $\theta > \widehat{\theta}$ together with $h(\theta) < x < g(\theta)$, then the associated condition (26) is violated. Again, $\varphi'(x) < 0$ for any degree of concavity of $K(\cdot)$. If $x \notin [h(\theta), g(\theta)]$, then the associated condition (25) or (26) can be satisfied. Because $\theta^+ > \theta > g(\theta)$, only θ^- can belong to $[h(\theta), g(\theta)]$. Hence, (24) is satisfied for any $\theta^+ < \theta$ if it is satisfied for $\theta^+ \to \theta$. Therefore, (24) is to be verified for $\theta^+ \to \theta$ and some $\theta^- < \theta$ (see here below).

Verify (24) on the range $[g(\theta), h(\theta)]$ when $\theta < \widehat{\theta}$ and $[h(\theta), g(\theta)]$ when $\theta > \widehat{\theta}$

(I) $\theta < \widehat{\theta}$ and $\theta^- \to \theta$.

$\varphi'\left(\theta^+\right) > 0$ if $\theta^+ < g(\theta)$; $\varphi'\left(\theta^+\right) < 0$ if $\theta^+ \in [g(\theta), h(\theta)]$; $\varphi'\left(\theta^+\right) > 0$ if $\theta^+ > h(\theta)$. Hence, (24) is tighter at θ than at $g(\theta)$; it is tighter at $h(\theta)$ than at $g(\theta)$; it is tighter at $h(\theta)$ than at $\overline{\theta}$. To verify if (24) is tightest as $\theta^+ \to \theta$ we need to recall the definition of $\varphi(x)$ and check whether:

$$\varphi(\theta) < \varphi(h(\theta)). \tag{47}$$

(47) is equivalent to:

$$\frac{q^*(\theta) + K'(\theta)}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}} < \frac{q^*(\theta) + \frac{K(\theta)-K(h(\theta))}{\theta - h(\theta)}}{\frac{p_1(\theta)-p_1(h(\theta))}{(\theta-h(\theta))p_1(\theta)} - \frac{p_n(\theta)-p_n(h(\theta))}{(\theta-h(\theta))p_n(\theta)}} = 0.$$

This is impossible because the left-hand side is positive. Hence, the left-hand side of (24) is highest for $\theta^+ = h(\theta)$, in which case $\varphi\left(\theta^+\right) = 0$. Because the right-hand side of (24) is lowest for $\theta^+ \to \theta$, we need to compare (24) for $\theta^+ = h(\theta)$ and for $\theta^+ \to \theta$. For $\theta^+ = h(\theta)$ and $\theta^- \to \theta$ (24) is rewritten as (27). For $\theta^+ \to \theta$ (24) reduces to $0 \leq 0$, and hence it is satisfied as an identity. Therefore, (24) holds for any triplet $\left\{\theta^-, \theta, \theta^+\right\}$ if and only if (27) is satisfied.

(II) $\theta > \widehat{\theta}$ and $\theta^+ \to \theta$.

$\varphi'\left(\theta^-\right) > 0$ if $\theta^- > g(\theta)$; $\varphi'\left(\theta^-\right) < 0$ if $\theta^- \in [h(\theta), g(\theta)]$; $\varphi'\left(\theta^-\right) > 0$ if $\theta^- < h(\theta)$. Hence, (24) is tighter at θ than at $g(\theta)$; it is tighter at $h(\theta)$ than at $g(\theta)$; it is tighter at $h(\theta)$ than at $\underline{\theta}$. To verify if (24) is tightest as $\theta^- \to \theta$ we need to check if:

$$\varphi(\theta) > \varphi(h(\theta)) \tag{48}$$

(48) is equivalent to:

$$\frac{q^*(\theta) + K'(\theta)}{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}} > \frac{q^*(\theta) + \frac{K(\theta)-K(h(\theta))}{\theta - h(\theta)}}{\frac{p_1(\theta)-p_1(h(\theta))}{(\theta-h(\theta))p_1(\theta)} - \frac{p_n(\theta)-p_n(h(\theta))}{(\theta-h(\theta))p_n(\theta)}} = 0,$$

which is not true because the left-hand side is negative. As in (I), we need to verify (24) for $\theta^- = h(\theta)$. Provided that $\theta^+ \to \theta$, (24) is rewritten as the converse of (27), namely as (28).

H Proof of Corollary 3

(27) implies (22)

Multiply both sides of (27) by the positive difference $\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}$ and rearrange to obtain

$$\begin{aligned} & q^*(\theta) + K'(\theta) \\ \leq \;& L\left\{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_1'(\theta)}{p_1(\theta)}p_n(\theta) + p_n'(\theta) - \frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}}{\frac{p_1(h(\theta))}{p_1(\theta)} - \frac{p_n(h(\theta))}{p_n(\theta)}}\left[(1-p_n(\theta))\frac{p_1(h(\theta))}{p_1(\theta)} - 1 + p_n(h(\theta))\right]\right\}. \end{aligned}$$

As the right-hand side is lower than $Lp_1'(\theta)/p_1(\theta)$, this is tighter than (22) if and only if

$$-\frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}}{\frac{p_1(h(\theta))}{p_1(\theta)} - \frac{p_n(h(\theta))}{p_n(\theta)}}\left[(1-p_n(\theta))\frac{p_1(h(\theta))}{p_1(\theta)} - 1 + p_n(h(\theta))\right] < p_n(\theta)\left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right).$$

Multiplying both sides by $\left[\left(\frac{p_1(h(\theta))}{p_1(\theta)} - \frac{p_n(h(\theta))}{p_n(\theta)}\right) \Big/ \left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right)\right] > 0$ and rearranging, we obtain $[(p_1(h(\theta)) - p_1(\theta))/p_1(\theta)] > 0$, which is true given (3).

(28) implies (23)

Multiply both sides of (28) by the positive difference $\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}$ and rearrange to obtain

$$\begin{aligned} & q^*(\theta) + K'(\theta) \\ \geq \;& L\left\{(1-p_n(\theta))\frac{p_1'(\theta)}{p_1(\theta)} + p_n'(\theta) - \frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}}{\frac{p_1(h(\theta))}{p_1(\theta)} - \frac{p_n(h(\theta))}{p_n(\theta)}}\left[(1-p_n(\theta))\frac{p_1(h(\theta))}{p_1(\theta)} - 1 + p_n(h(\theta))\right]\right\}. \end{aligned}$$

As the right-hand side is higher than $Lp_n'(\theta)/p_n(\theta)$, this is tighter than (23) if and only if

$$-\frac{\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}}{\frac{p_n(h(\theta))}{p_n(\theta)} - \frac{p_1(h(\theta))}{p_1(\theta)}}\left[(1-p_n(\theta))\frac{p_1(h(\theta))}{p_1(\theta)} - 1 + p_n(h(\theta))\right] < (1-p_n(\theta))\left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right).$$

Multiplying both sides by $\left[\left(\frac{p_n(h(\theta))}{p_n(\theta)} - \frac{p_1(h(\theta))}{p_1(\theta)}\right) \Big/ \left(\frac{p_1'(\theta)}{p_1(\theta)} - \frac{p_n'(\theta)}{p_n(\theta)}\right)\right] > 0$ and rearranging, we obtain $[(p_n(h(\theta)) - p_n(\theta))/p_n(\theta)] > 0$, which is true given (3).