Prova scritta di Matematica per l'Economia L-Z- 20 Dicembre 2021

1. Studiare e tracciare il grafico della funzione:

$$f(x) = \ln^2 x - 5\ln x + 6$$

2. Approssimare con il polinomio di Taylor di grado $n = 2$ e punto iniziale $x_0 = 0$ la funzione:

$$f(x) = \sqrt{1 + \sin x}$$

3. Calcolare il seguente integrale:

$$\int \frac{-2x^3 + 3x^2 + 3x - 1}{x^2 - x - 2} dx$$

4. Studiare il sistema $Ax = b$ al variare del parametro $k \in \mathbb{R}$:

$$A = \begin{pmatrix} 0 & 4 & 1 \\ -2k & 2 & -3 \\ 1 & -2 & 1 \end{pmatrix}; \quad b = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}; \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix};$$

SVOLGIMENTO

1) Dominio $D =]0, +\infty[$

Positività; $f(x) > 0$ $\quad \ln^2 x - 5\ln x + 6 > 0$

$\ln x = y$

$$y^2 - 5y + 6 > 0 \quad y_{1,2} = \frac{5 \pm \sqrt{25-24}}{2}$$

$$y_1 = \frac{5+1}{2} = 3 \; ; \; y_2 = \frac{5-1}{2} = 2$$

$y < 2 \Rightarrow \ln x < 2 \quad x < e^2$

$y > 3 \Rightarrow \ln x > 3 \quad x > e^3$

$f(x)>0 \quad]0, e^2[\cup]e^3, +\infty[$

$f(x)=0 \quad x=e^2;\ x=e^3$

$f(x)<0 \quad]e^2, e^3[$

$x=0$ ASINT. VERT

limiti:

$$\lim_{x\to 0^+} \ln^2 x - 5\ln x + 6 = +\infty + \infty = +\infty$$

$$\lim_{x\to +\infty} \ln^2 x - 5\ln x + 6 = +\infty - \infty = ??$$

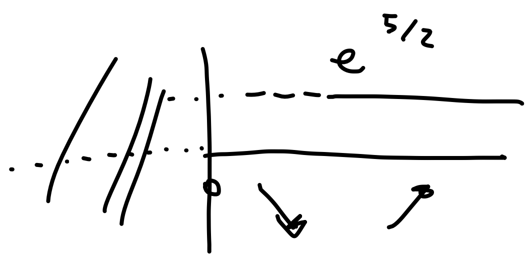
$$= \ln^2 x \left(1 - \frac{5}{\ln x} + \frac{6}{\ln^2 x}\right) = +\infty$$

$\frac{5}{\infty} = 0 \qquad \to 0 \quad \frac{6}{\infty} = 0$

Derivate prime

$$f'(x) = \frac{2\ln x}{x} - \frac{5}{x} = \frac{2\cdot \ln x - 5}{x}$$

$f'(x)>0 \quad \frac{2\ln x - 5}{x} > 0 \quad \begin{cases} 2\ln x - 5 > 0 \Rightarrow \ln x > \frac{5}{2} \quad x > e^{5/2} \\ x>0 \end{cases}$



$f'(x)>0 \quad]e^{5/2}, +\infty[$

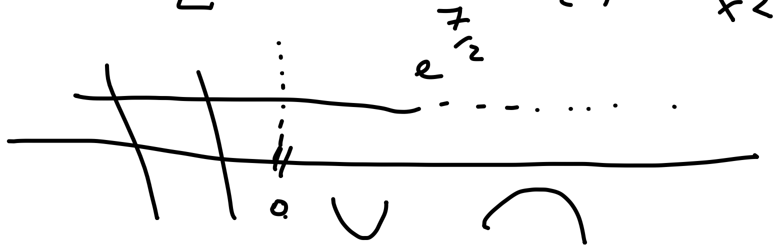
$f'(x)=0 \quad x=e^{5/2}$ (P.to minimo relativo)

$f'(x)<0 \quad]0, e^{5/2}[$

Derivate seconde

$$f''(x) = \frac{\frac{2}{x}\cdot x - (2\ln x - 5)}{x^2} = \frac{2 - 2\ln x + 5}{x^2} = \frac{7 - 2\ln x}{x^2}$$

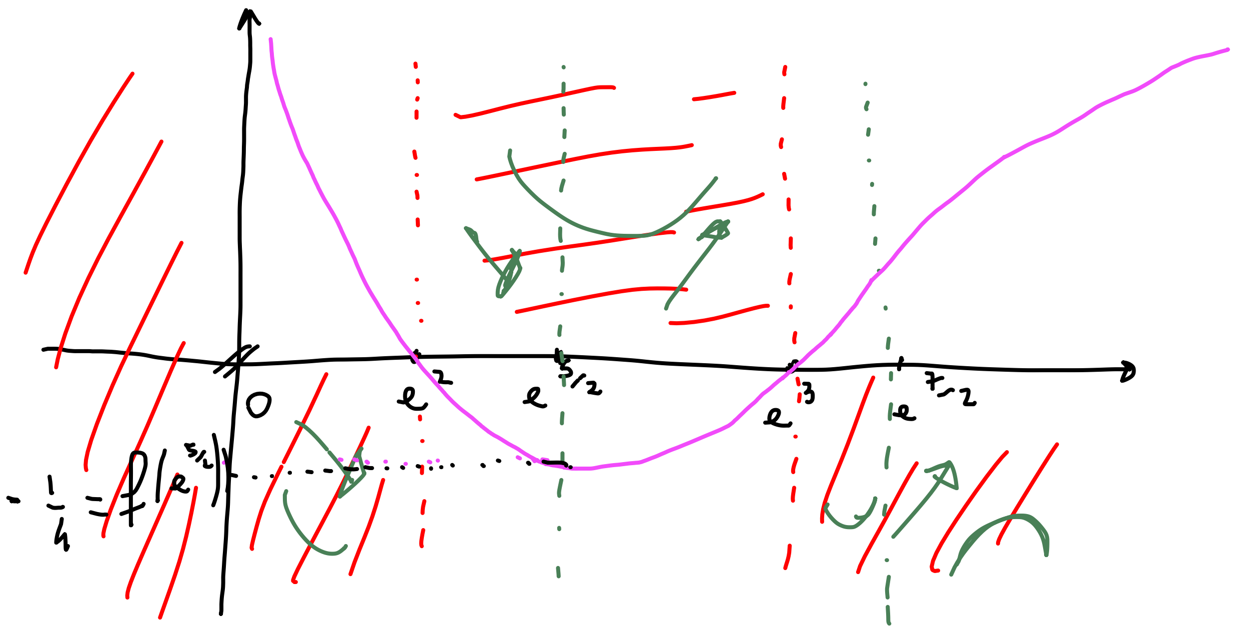
$$f''(x) > 0 \quad \frac{7-2\ln x}{x^2} > 0 \quad \left[\begin{array}{l} 7-2\ln x > 0 \Rightarrow \ln x < \frac{7}{2} \\ x^2 > 0 \quad \forall x \in \mathbb{R}-\{0\} \end{array}\right. \quad x < e^{7/2}$$



$$f''(x) > 0 \quad]0, e^{7/2}[$$

$$f''(x) = 0 \quad x = e^{7/2} \text{ (Flesso)}$$

$$f''(x) < 0 \quad]e^{7/2}, +\infty[$$



f BIUNIVOCA? NO

codominio $[f(e^{5/2}), +\infty[= [-\frac{1}{4}, +\infty[$

$$f(e^{5/2}) = (\ln e^{5/2})^2 - 5\ln e^{5/2} + 6$$

$$= \left(\frac{5}{2}\right)^2 - 5\cdot\frac{5}{2} + 6 = \frac{25}{4} - \frac{25}{2} + 6$$

$$= \frac{25-50+24}{4} = -\frac{1}{4}$$

$m = -\frac{1}{4}$ minimo assoluto

2) Taylor $f(x)=\sqrt{1+\operatorname{sen} x}$; $f(0)=1$

$$f'(x)=\frac{1}{2\sqrt{1+\operatorname{sen} x}}\cdot\cos x \; ; \; f'(0)=\frac{1}{2}$$

$$f''(x)=\frac{1}{2}\left[\frac{-\operatorname{sen} x\cdot\sqrt{1+\operatorname{sen} x}-\cos x\cdot\frac{\cos x}{2\sqrt{1+\operatorname{sen} x}}}{\left(\sqrt{1+\operatorname{sen} x}\right)^2}\right]\cdot$$

$$=\frac{1}{2}\left[\frac{-2\operatorname{sen} x\,(1+\operatorname{sen} x)-\cos^2 x}{\sqrt{1+\operatorname{sen} x}\cdot 2(1+\operatorname{sen} x)}\right]=\frac{1}{2}\left[\frac{-2\operatorname{sen} x-2\operatorname{sen}^2 x-\cos^2 x}{\sqrt{1+\operatorname{sen} x}\cdot 2(1+\operatorname{sen} x)}\right]$$

$$f''(0)=-\frac{1}{4}$$

$$\sqrt{1+\operatorname{sen} x}\approx 1+\frac{1}{2}x-\frac{1}{4}\frac{x^2}{2}=1+\frac{x}{2}-\frac{x^2}{8}$$

3)

$$\begin{array}{rrrr|l}
-2x^3 & +3x^2 & +3x & -1 & x^2-x-2 \\
-2x^3 & +2x^2 & +4x & & -2x+1 \\
\hline
// & x^2 & -x & -1 & \\
 & x^2 & -x & -2 & \\
\hline
 & // & // & 1 &
\end{array}$$

$$\int\frac{-2x^3+3x^2+3x-1}{x^2-x-2}\,dx=\int -2x+1+\frac{1}{x^2-x-2}\,dx=-\frac{2x^2}{2}+x$$

$$+\int\frac{1}{(x^2-x-2)}\,dx \qquad x^2-x-2= \frac{1\pm\sqrt{1+8}}{2} \begin{cases}\frac{1+3}{2}=2\\ \frac{1-3}{2}=-1\end{cases}$$

$$\Rightarrow \quad \frac{1}{(x-2)(x+1)}=\frac{A_1}{x-2}+\frac{A_2}{x+1}$$

$$\frac{A_1(x+1)+A_2(x-2)}{(x-2)(x+1)} = \frac{x(A_1+A_2)+A_1-2A_2}{(x-2)(x+1)}$$

$$\begin{cases} A_1+A_2=0 \\ A_1-2A_2=1 \end{cases} \quad \begin{cases} A_1=-A_2 \\ -A_2-2A_2=1 \Rightarrow A_2=-\frac{1}{3} \end{cases}$$

$$A_1=\frac{1}{3}$$

$$\int \frac{1}{(x^2-x-2)}dx = \int \frac{1/3}{(x-2)}dx + \int \frac{-1/3}{(x+1)}dx$$

$$=\frac{1}{3}\ln|x-2|-\frac{1}{3}\ln|x+1|$$

$$I = -x^2+x+\frac{1}{3}\ln\left|\frac{x-2}{x+1}\right|$$

4) Sistema

$$\det(A) = 4\cdot C_{12}+1\cdot C_{13} = 4\cdot(2k-3)-2+4k = 8k+4k-12-2$$
$$=12k-14$$

$$C_{12}=(-1)^3\cdot\begin{vmatrix} -2k & -3 \\ 1 & 1 \end{vmatrix} = -1(-2k+3) = 2k-3$$

$$C_{13}=(-1)^4\begin{vmatrix} -2k & 2 \\ 1 & -2 \end{vmatrix} = (+4k-2)$$

Se $\det(A)\neq 0 \Rightarrow k \neq \frac{14}{12}=\frac{7}{6}$ $\operatorname{car}(A)=3$ e $\operatorname{car}(B)=3$

e il sistema ammette una sola soluzione

$$x_1 = \frac{\begin{vmatrix} 0 & 4 & 1 \\ 2 & 2 & -3 \\ 1 & -2 & 1 \end{vmatrix}}{4k-14} = -4\cdot\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} + 1\cdot\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} = -4(2+3)+(-4-2)$$

$$=\frac{-20-6}{4k-14} = \frac{-26}{12k-14}$$

$$x_2 = \frac{\begin{vmatrix} 0 & 0 & 1 \\ -2k & 2 & -3 \\ 1 & 1 & 1 \end{vmatrix}}{4k-14} = \frac{+1 \cdot \begin{vmatrix} -2k & 2 \\ 1 & 1 \end{vmatrix}}{4k-14} = \frac{+(-2k-2)}{4k-14} = \frac{-2k-2}{12k-14}$$

$$x_3 = \frac{\begin{vmatrix} 0 & 4 & 0 \\ -2k & 2 & 2 \\ 1 & -2 & 1 \end{vmatrix}}{4k-14} = \frac{-4 \begin{vmatrix} -2k & 2 \\ 1 & 1 \end{vmatrix}}{4k-14} = \frac{-4(-2k-2)}{4k-14} =$$

$$\frac{+8k+8}{12k-14}$$

$\det \neq 0$

Se $k = \frac{7}{2}$ allora $A = \begin{pmatrix} 0 & 4 & 1 \\ -7 & 2 & -3 \\ 1 & -2 & 1 \end{pmatrix}$;

$car(A) = 2$

$B = \begin{pmatrix} 0 & 4 & 1 & 0 \\ -7 & 2 & -3 & 2 \\ 1 & -2 & 1 & 1 \end{pmatrix}$; $B^* = \begin{pmatrix} 0 & 4 & 0 \\ -7 & 2 & 2 \\ 1 & -2 & 1 \end{pmatrix}$

$\det(B^*) = -4 \begin{vmatrix} -7 & 2 \\ 1 & 1 \end{vmatrix} = -4 \cdot (-7-2) = -4 \cdot -9 = 36 \neq 0$

$car(B) = 3$ e il sistema è impossibile.