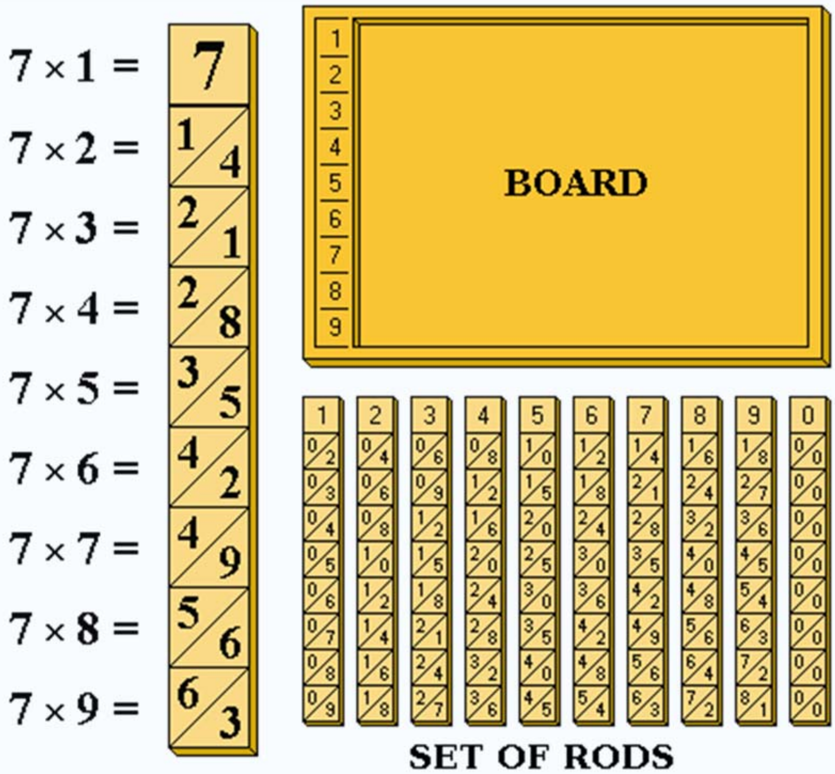
Napier's bones

Napier's bones are an abacus invented by John Napier for calculation of products and quotients of numbers. Also called **Rabdology** (from Greek ραβδος [rabdos], rod and λόγος [logos], word). Napier published his invention of the rods in a work printed in Edinburgh, Scotland, at the end of 1617 entitled *Rabdologiæ*. Using the multiplication tables embedded in the rods, multiplication can be reduced to addition operations and division to subtractions. More advanced use of the rods can even extract square roots. Note that **Napier's bones** are not the same as logarithms, with which Napier's name is also associated.



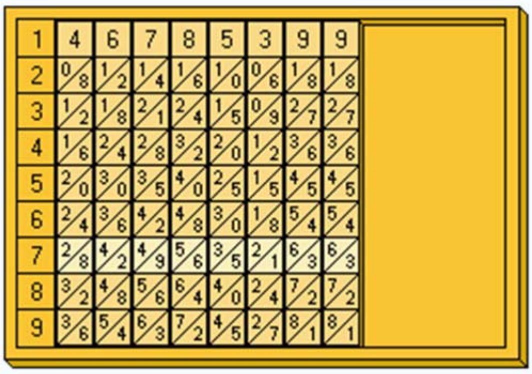
The abacus consists of a board with a rim; the user places Napier's rods in the rim to conduct multiplication or division. The board's left edge is divided into 9 squares, holding the numbers 1 to 9. The **Napier's rods** consist of strips of wood, metal or heavy cardboard. **Napier's bones** are three dimensional, square in cross section, with four different **rods** engraved on each one. A set of such **bones** might be enclosed in a convenient carrying case.

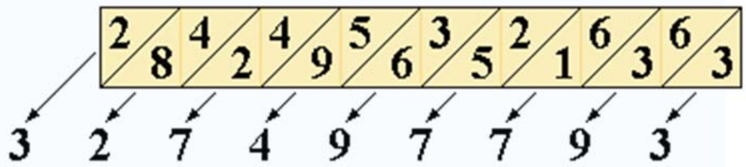
A rod's surface comprises 9 squares, and each square, except for the top one, comprises two halves divided by a diagonal line. The first square of each rod holds a single-digit, and the other squares hold this number's double, triple, quadruple and so on until the last square contains nine times the number in the top square. The digits of each product are written one to each side of the diagonal; numbers less than 10 occupy the lower triangle, with a zero in the top half.

A set consists of 9 rods corresponding to digits 1 to 9. The figure additionally shows the rod 0; although for obvious reasons it is not necessary for calculations.

Multiplication

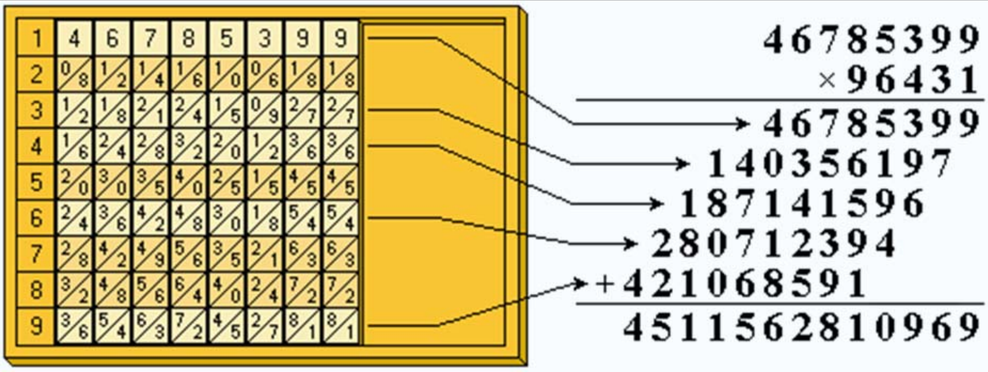
Given the described set of rods, suppose that we wish to calculate the product of **46785399** and **7**. Place inside the board the rods corresponding to 46785399, as shown in the diagram, and read the result in the horizontal strip in row 7, as marked on the side of the board. To obtain the product, simply note, for each place from right to left, the numbers found by adding the digits within the diagonal sections of the strip (using carry-over where the sum is 10 or greater).





From right to left, we obtain the units place (3), the tens (6+3=9), the hundreds (6+1=7), etc. Note that in the hundred thousands place, where 5+9=14, we note '4' and carry '1' to the next addition (similarly with 4+8=12 in the ten millions place).

In cases where a digit of the multiplicand is 0, we leave a space between the rods corresponding to where a 0 rod would be. Let us suppose that we want to multiply the previous number by **96431**; operating analogously to the previous case, we will calculate partial products of the number by multiplying 46785399 by 9, 6, 4, 3 and 1. Then we place these products in the appropriate positions, and add them using the simple pencil-and-paper method.



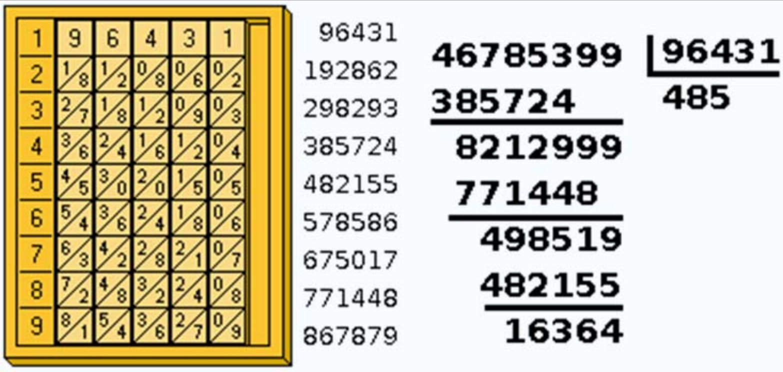
This method can also be used for multiplying decimals. For a decimal value multiplied by an integer (whole number) value ensure that the decimal number is written along the top of the grid. From this position the decimal point simply drops down the vertical line and 'falls' into the answer.

When multiplying two decimal numbers together, the decimal points travel horizontally and vertically until they 'meet' at a diagonal line, the point then travels out of the grid in the same method and again 'falls' into the answer.

The form of multiplication was also used in the 1202 Liber Abaci and 800 AD Islamic mathematics and known under the name of lattice multiplication. "Crest of the Peacock", by G.G, Joseph, suggests that Napier learned the details of this method from "Treviso Arithmetic", written in 1478.

Division

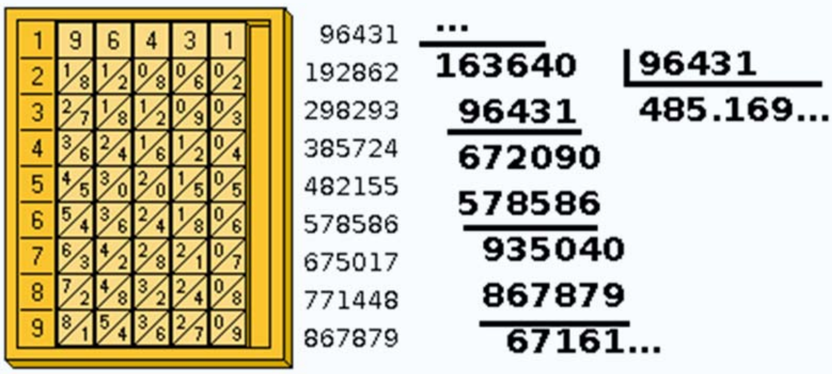
Division can be performed in a similar fashion. Let's divide 46785399 by 96431, the two numbers we used in the earlier example. Put the bars for the divisor (96431) on the board, as shown in the graphic below. Using the abacus, find all the products of the divisor from 1 to 9 by reading the displayed numbers. Note that the dividend has eight digits, whereas the partial products (save for the first one) all have six. So you must temporarily ignore the final two digits of 46785399, namely the '99', leaving the number 467853. Next, look for the greatest partial product that is less than the truncated dividend. In this case, it's 385724. You must mark down two things, as seen in the diagram: since 385724 is in the '4' row of the abacus, mark down a '4' as the left-most digit of the quotient; also write the partial product, left-aligned, under the original dividend, and subtract the two terms. You get the difference as 8212999. Repeat the same steps as above: truncate the number to six digits, chose the partial product immediately less than the truncated number, write the row number as the next digit of the quotient, and subtract the partial product from the difference found in the first repetition. Following the diagram should clarify this. Repeat this cycle until the result of subtraction is less than the divisor. The number left is the remainder.



So in this example, we get a quotient of 485 with a remainder of 16364. We can just stop here and use the fractional form of the answer $485\frac{16364}{96431}$.

If you prefer, we can also find as many decimal points as we need by continuing the cycle as in standard long division. Mark a decimal point after the last digit of the quotient and append a zero to the remainder so we now have 163640. Continue the cycle, but each time appending a zero to the result after the subtraction.

Let's work through a couple of digits. The first digit after the decimal point is 1, because the biggest partial product less than 163640 is 96431, from row 1. Subtracting 96431 from 163640, we're left with 67209. Appending a zero, we have 672090 to consider for the next cycle (with the partial result 485.1) The second digit after the decimal point is 6, as the biggest partial product less than 672090 is 578586 from row 6. The partial result is now 485.16, and so on.



Extracting square roots

Extracting the square root uses an additional bone which looks a bit different from the others as it has three columns on it. The first column has the first nine squares 1, 4, 9, ... 64, 81, the second column has the even numbers 2 through 18, and the last column just has the numbers 1 through 9.

Napier's rods with the square root bone									
1	2	3	4	5	6	7	8	9	√

1	$^0/_1$	$^0/_2$	$^0/_3$	$^0/_4$	$^0/_5$	$^0/_6$	$^0/_7$	$^0/_8$	$^0/_9$	$^0/_1$	2	1
2	$^0/_2$	$^0/_4$	$^0/_6$	$^0/_8$	$^1/_0$	$^1/_2$	$^1/_4$	$^1/_6$	$^1/_8$	$^0/_4$	4	2
3	$^0/_3$	$^0/_6$	$^0/_9$	$^1/_2$	$^1/_5$	$^1/_8$	$^2/_1$	$^2/_4$	$^2/_7$	$^0/_9$	6	3
4	$^0/_4$	$^0/_8$	$^1/_2$	$^1/_6$	$^2/_0$	$^2/_4$	$^2/_8$	$^3/_2$	$^3/_6$	$^1/_6$	8	4
5	$^0/_5$	$^1/_0$	$^1/_5$	$^2/_0$	$^2/_5$	$^3/_0$	$^3/_5$	$^4/_0$	$^4/_5$	$^2/_5$	10	5
6	$^0/_6$	$^1/_2$	$^1/_8$	$^2/_4$	$^3/_0$	$^3/_6$	$^4/_2$	$^4/_8$	$^5/_4$	$^3/_6$	12	6
7	$^0/_7$	$^1/_4$	$^2/_1$	$^2/_8$	$^3/_5$	$^4/_2$	$^4/_9$	$^5/_6$	$^6/_3$	$^4/_9$	14	7
8	$^0/_8$	$^1/_6$	$^2/_4$	$^3/_2$	$^4/_0$	$^4/_8$	$^5/_6$	$^6/_4$	$^7/_2$	$^6/_4$	16	8
9	$^0/_9$	$^1/_8$	$^2/_7$	$^3/_6$	$^4/_5$	$^5/_4$	$^6/_3$	$^7/_2$	$^8/_1$	$^8/_1$	18	9

Let's find the square root of 46785399 with the bones.

First, group its digits in twos starting from the right so it looks like this:

46 78 53 99

Note: A number like 85399 would be grouped as 8 53 99

Start with the leftmost group 46. Pick the largest square on the square root bone less than 46, which is 36 from the sixth row.

Because we picked the sixth row, the first digit of the solution is 6. Now read the second column from the sixth row on the square root bone, 12, and set 12 on the board. Then subtract the value in the first column of the sixth row, 36, from 46. Append to this the next group of digits in the number 78, to get the remainder 1078.

At the end of this step, the board and intermediate calculations should look like this:

	1	**2**	√		
1	$^0/_1$	$^0/_2$	$^0/_1$	2	1
2	$^0/_2$	$^0/_4$	$^0/_4$	4	2
3	$^0/_3$	$^0/_6$	$^0/_9$	6	3
4	$^0/_4$	$^0/_8$	$^1/_6$	8	4
5	$^0/_5$	$^1/_0$	$^2/_5$	10	5
6	$^0/_6$	$^1/_2$	$^3/_6$	12	6
7	$^0/_7$	$^1/_4$	$^4/_9$	14	7
8	$^0/_8$	$^1/_6$	$^6/_4$	16	8
9	$^0/_9$	$^1/_8$	$^8/_1$	18	9

```
  _____________
 √46 78 53 99    =     6
  36
  --
  10 78
```

Now "read" the number in each row (ignore the second and third columns from the square root bone.) For example, read the sixth row as

$^0/_6\ ^1/_2\ ^3/_6 \rightarrow 756$

Now find the largest number less than the current remainder, 1078. You should find that 1024 from the eighth row is the largest value less than 1078.

	1	2	√			(value)
1	0/1	0/2	0/1	2	1	*121*
2	0/2	0/4	0/4	4	2	*244*
3	0/3	0/6	0/9	6	3	*369*
4	0/4	0/8	1/6	8	4	*496*
5	0/5	1/0	2/5	10	5	*625*
6	0/6	1/2	3/6	12	6	*756*
7	0/7	1/4	4/9	14	7	*889*
8	0/8	1/6	6/4	16	8	*1024*
9	0/9	1/8	8/1	18	9	*1161*

```
          ______________
          √46 78 53 99
=    68
          36
          --
          10 78
          10 24
          -----
             54
```

As before, append 8 to get the next digit of the square root and subtract the value of the eighth row 1024 from the current remainder 1078 to get 54. Read the second column of the eighth row on the square root bone, 16, and set the number on the board as follows.

The current number on the board is 12. Add to it the first digit of 16, and append the second digit of 16 to the result. So you should set the board to

12 + 1 = 13 → append 6 → 136

Note: If the second column of the square root bone has only one digit, just append it to the current number on board.

The board and intermediate calculations now look like this.

	1	3	6	√		
1	0/1	0/3	0/6	0/1	2	1
2	0/2	0/6	1/2	0/4	4	2
3	0/3	0/9	1/8	0/9	6	3
4	0/4	1/2	2/4	1/6	8	4
5	0/5	1/5	3/0	2/5	10	5
6	0/6	1/8	3/6	3/6	12	6
7	0/7	2/1	4/2	4/9	14	7
8	0/8	2/4	4/8	6/4	16	8
9	0/9	2/7	5/4	8/1	18	9

```
 ______________
 √46 78 53 99    =     68
  36
  --
  10 78
  10 24
  -----
     54 53
```

	1	3	6	√	

1	0/1	0/3	0/6	0/1	2	1	*1361*
2	0/2	0/6	1/2	0/4	4	2	*2724*
3	0/3	0/9	1/8	0/9	6	3	*4089*
4	0/4	1/2	2/4	1/6	8	4	*5456*
5	0/5	1/5	3/0	2/5	10	5	*6825*
6	0/6	1/8	3/6	3/6	12	6	*8196*
7	0/7	2/1	4/2	4/9	14	7	*9569*
8	0/8	2/4	4/8	6/4	16	8	*10944*
9	0/9	2/7	5/4	8/1	18	9	*12321*

Once again, find the row with the largest value less than the current partial remainder 5453. This time, it is the third row with 4089.

```
 _____________
      √46 78 53 99  = 68**3**
        36
        --
        10 78
        10 24
        -----
        54 53
        **40 89** -
        -----
        **13 64**
```

The next digit of the square root is 3. Repeat the same steps as before and subtract 4089 from the current remainder 5453 to get 1364 as the next remainder. When you rearrange the board, notice that the second column of the square root bone is 6, a single digit. So just append 6 to the current number on the board 136

136 → append 6 → 1366

to set 1366 on the board.

	1	3	6	6	√	
1	0/1	0/3	0/6	0/6	0/1	2 1
2	0/2	0/6	1/2	1/2	0/4	4 2
3	0/3	0/9	1/8	1/8	0/9	6 3
4	0/4	1/2	2/4	2/4	1/6	8 4
5	0/5	1/5	3/0	3/0	2/5	10 5
6	0/6	1/8	3/6	3/6	3/6	12 6
7	0/7	2/1	4/2	4/2	4/9	14 7
8	0/8	2/4	4/8	4/8	6/4	16 8
9	0/9	2/7	5/4	5/4	8/1	18 9

```
 _____________
      √46 78 53 99  = 683
        36
        --
        10 78
        10 24
        -----
```

```
-

54 53

40 89
                -
----

13 64 99
```

Repeat these operations once more. Now the largest value on the board smaller than the current remainder 136499 is 123021 from the ninth row.

In practice, you often don't need to find the value of every row to get the answer. You may be able to guess which row has the answer by looking at the number on the first few bones on the board and comparing it with the first few digits of the remainder. But in these diagrams, we show the values of all rows to make it easier to understand.

As usual, append a 9 to the result and subtract 123021 from the current remainder.

	1	3	6	6	√		
1	0/1	0/3	0/6	0/6	0/1	2 1	*13661*
2	0/2	0/6	1/2	1/2	0/4	4 2	*27324*
3	0/3	0/9	1/8	1/8	0/9	6 3	*40989*
4	0/4	1/2	2/4	2/4	1/6	8 4	*54656*
5	0/5	1/5	3/0	3/0	2/5	10 5	*68325*
6	0/6	1/8	3/6	3/6	3/6	12 6	*81996*
7	0/7	2/1	4/2	4/2	4/9	14 7	*95669*
8	0/8	2/4	4/8	4/8	6/4	16 8	*109344*
9	0/9	2/7	5/4	5/4	8/1	18 9	*123021*

```
_____________
          √46
78 53 99     =
6839
          36
          --
          10
78
          10
24
          ----
-

54 53

40 89
             -
----

13 64 99

12 30 21
             -
-------
```

```
1 34 78
```

You've now "used up" all the digits of our number, and you still have a remainder. This means you've got the integer portion of the square root but there's some fractional bit still left.

Notice that if we've really got the integer part of the square root, the current result squared ($6839^2 = 46771921$) must be the largest perfect square smaller than 46785899. Why? The square root of 46785399 is going to be something like 6839.xxxx... This means 6839^2 is smaller than 46785399, but 6840^2 is bigger than 46785399 -- the same thing as saying that 6839^2 is the largest perfect square smaller than 46785399.

This idea is used later on to understand how the technique works, but for now let's continue to generate more digits of the square root.

Similar to finding the fractional portion of the answer in long division, append two zeros to the remainder to get the new remainder 1347800. The second column of the ninth row of the square root bone is 18 and the current number on the board is 1366. So compute

1366 + 1 → 1367 → append 8 → 13678

to set 13678 on the board.

The board and intermediate computations now look like this.

	1	3	6	7	8	√		
1	0/1	0/3	0/6	0/7	0/8	0/1	2	1
2	0/2	0/6	1/2	1/4	1/6	0/4	4	2
3	0/3	0/9	1/8	2/1	2/4	0/9	6	3
4	0/4	1/2	2/4	2/8	3/2	1/6	8	4
5	0/5	1/5	3/0	3/5	4/0	2/5	10	5
6	0/6	1/8	3/6	4/2	4/8	3/6	12	6
7	0/7	2/1	4/2	4/9	5/6	4/9	14	7
8	0/8	2/4	4/8	5/6	6/4	6/4	16	8
9	0/9	2/7	5/4	6/3	7/2	8/1	18	9

```
______________
           √46
78 53 99      =
6839.
           36
           --
           10
78
           10
24
           ----
-
```

```
54 53

40 89
                       -
----

13 64 99

12 30 21
                       -
-------

1 34 78 00
```

The ninth row with 1231101 is the largest value smaller than the remainder, so the first digit of the fractional part of the square root is 9.

	1	3	6	7	8	√			
1	0/1	0/3	0/6	0/7	0/8	0/1	2	1	*136781*
2	0/2	0/6	1/2	1/4	1/6	0/4	4	2	*273564*
3	0/3	0/9	1/8	2/1	2/4	0/9	6	3	*410349*
4	0/4	1/2	2/4	2/8	3/2	1/6	8	4	*547136*
5	0/5	1/5	3/0	3/5	4/0	2/5	10	5	*683925*
6	0/6	1/8	3/6	4/2	4/8	3/6	12	6	*820716*
7	0/7	2/1	4/2	4/9	5/6	4/9	14	7	*957509*
8	0/8	2/4	4/8	5/6	6/4	6/4	16	8	*1094304*
9	0/9	2/7	5/4	6/3	7/2	8/1	18	9	*1231101*

```
 __________________
                √46
78 53 99        =
6839.9
                 36
                 --
                 10
78
                 10
24
                 ----
-

54 53

40 89
                     -
----

13 64 99

12 30 21
                     -
-------

1 34 78 00

1 23 11 01

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```

```
11 66 99
```

Subtract the value of the ninth row from the remainder and append a couple more zeros to get the new remainder 11669900. The second column on the ninth row is 18 with 13678 on the board, so compute

13678 + 1 → 13679 → append 8 → 136798

and set 136798 on the board.

	1	3	6	7	9	8	√	
1	0/1	0/3	0/6	0/7	0/9	0/8	0/1	2 1
2	0/2	0/6	1/2	1/4	1/8	1/6	0/4	4 2
3	0/3	0/9	1/8	2/1	2/7	2/4	0/9	6 3
4	0/4	1/2	2/4	2/8	3/6	3/2	1/6	8 4
5	0/5	1/5	3/0	3/5	4/5	4/0	2/5	10 5
6	0/6	1/8	3/6	4/2	5/4	4/8	3/6	12 6
7	0/7	2/1	4/2	4/9	6/3	5/6	4/9	14 7
8	0/8	2/4	4/8	5/6	7/2	6/4	6/4	16 8
9	0/9	2/7	5/4	6/3	8/1	7/2	8/1	18 9

```
              √46
78 53 99     =
6839.9
              36
              --
              10
78
              10
24
              ----
-

54 53

40 89
                -
----

13 64 99

12 30 21
                -
-------

1 34 78 00

1 23 11 01

----------

11 66 99 00
```

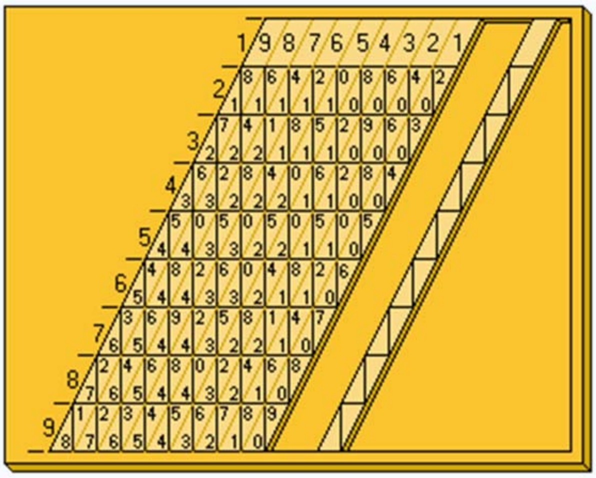
You can continue these steps to find as many digits as you need and you stop when you have the precision you want, or if you find that the reminder becomes zero which means you have the exact square root.

There's only one more trick left to describe. If you want to find the square root of a number that isn't an integer, say 54782.917. Everything is the same, except you start out by grouping the digits to the left and right of the decimal point in groups of two.

That is, group 54782.917 as 5 47 82 . 91 7

and proceed to extract the square root from these groups of digits.

Modifications



During the 19th century, Napier's bones underwent a transformation to make them easier to read. The rods began to be made with an angle of about 65° so that the triangles that had to be added were aligned vertically. In this case, in each square of the rod the unit is to the right and the ten (or the zero) to the left.

The rods were made such that the vertical and horizontal lines were more visible than the line where the rods touched, making the two components of each digit of the result much easier to read. Thus, in the picture it is immediately clear that:

987654321 x 5 = 4938271605